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Research Article

Interacting Rényi Holographic Dark Energy in the Brans-Dicke Theory

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In this work, we construct an interacting model of the Rényi holographic dark energy in the Brans-Dicke theory of gravity using Rényi entropy in a spatially flat Friedmann-Lemaître-Robertson-Walker Universe considering the infrared cut-off as the Hubble horizon. In this setup, we then study the evolutionary history of some important cosmological parameters, in particular, deceleration parameter, Hubble parameter, equation of state parameter, and Rényi holographic dark energy density parameter in both nonflat Universe and flat Universe scenarios and also observe satisfactory behaviors of these parameters in the model. We find that during the evolution, the present model can give rise to a late-time accelerated expansion phase for the Universe preceded by a decelerated expansion phase for both flat and nonflat cases. Moreover, we obtain $\omega_D \to -1$ as $z \to -1$, which indicates that this model behaves like the cosmological constant at the future. The stability analysis for the distinct estimations of the Rényi parameter δ and coupling coefficient b^2 has been analyzed. The results indicate that the model is stable at the late time.

1. Introduction

The Virial theorem (1930s) which provided the Coma galaxy cluster mass [1, 2], accompanied by the galaxy rotation curve study (1970) [3] and the two different research groups' observational results in the 1990s [4, 5], has uncovered one of the most interesting issues of cosmology at present: the dark sector. It is suggested by the researchers that the five percent of the present energy content of the cosmos is composed of the radiation and the ordinary matter (baryons); the remaining ninety-five percent is dominated by this dark component to clarify the late accelerated expansion of the Universe. It is believed that this dark sector of the Universe mainly includes two constituents: *dark energy* (DE) and *dark matter* (DM). Both are important and significant to understand the phenomena of scales and nature. The significance of DM lies primarily in the structure formation, for instance, to permit baryonic structures to become nonlinear in the wake of decoupling from the photons. Interestingly, dark energy is the subject of study to answer the late-time accelerated expansion for the observable Universe [6]. Also, the DM is narrated as *cold dark matter* (CDM), and dark energy is portrayed by the *cosmological constant* (Λ) in the standard cosmological scenario. The dark component of the Universe with radiation and baryons combined the ΛCDM model. Also, despite the fact that the ΛCDM model appreciates an impressive observational achievement [7–9], there are still a number of hypothetical and observational focuses that have the right to be completely researched [10]. The greatest test lies in understanding the crucial idea of these dark sectors from the theoretical perspective [6]. In 2004, Li [11] proposed the idea of *holographic dark energy* (HDE) which is also used to explain the DE scenario to explain the late-time accelerated expansion of the Universe inspired by the holographic principle [12–18]. Right after a paper by Li, the most complete generalization which includes all known HDE models were suggested [19]. Furthermore, it is shown that the Nojiri-Odintsov HDE describes also covariant theories unlike Li's HDE [20].

Recently, inspired by the holographic principle and using the Rényi entropy [21], a new dark energy model has been proposed by Moradpour et al. [22] named the *Rényi holographic dark energy* (RHDE) model for the cosmological and gravitational investigations. Generalizing one of the entropy or gravity, as entropy-area connection relies on the gravity hypothesis, will change the corresponding one. It is proposed that by using the Rényi entropy, the modified Friedmann equations can be obtained [23–25]. Ghaffari et al. [26] proposed that inflation may be found in the Rényi formalism. The RHDE models have been explored with IR cut-off as the particle and future event horizons [27]. The spatially homogeneous and anisotropic Bianchi VI_0 Universe filled with RHDE with Granda-Oliveros and Hubble horizons as the IR cut-off has been investigated in general relativity [28]. Recently, Sharma et al. [29, 30] discriminated the RHDE model from the ΛCDM model by using different diagnostic tools such as statefinder diagnostic and statefinder hierarchy in ample details. Also, the RHDE model has been compared with the holographic and Tsallis holographic dark energy through the statefinder diagnostic tool [31].

Indeed, all the above attempts claim that, at least mathematically, the DE density profile introduced under the shadow of the RHDE hypothesis has considerable potential for modelling the DE behavior, and thus, more studies on this density profile are motivated. Further, at large scales, the models presenting interaction fare well when confronted with observational outcomes from the CMB [32] and matter distribution [33]. Therefore, the interaction between DE and DM must be handled seriously. Then again, there exist limits for the quality of this association for different setups [34–48]. This newly proposed Rényi HDE has also been examined by many researchers by considering the interaction between DE and DM to explain the accelerated expansion of the Universe with different IR cut-offs in general relativity, braneworld, loop quantum cosmology, and modified gravity [49–52]. Sharma and Dubey [53] investigated the Rényi HDE model in the Friedmann-Lemaître-Robertson-Walker (FLRW) Universe considering different parametrizations of the interaction between the DM and DE.

On the other side, the modified gravity theories have been broadly applied in cosmology [54–56]. The modified theories of gravity are not new and have a long history. A well-known modified gravity theory, the Brans-Dicke gravity [57], is also a choice to general relativity to explain the accelerated expansion of the cosmos [58] and also can pass the experimental tests from the solar system [59]. Theoretically, the value of the Brans-Dicke coupling parameter has a smaller value than observed by the observational data, which encouraged physicists to suggest various DE scenarios to describe the present cosmos in the Brans-Dicke formalism [58–60]. Using the holographic principle, Gong [61] investigated the holographic bound in the Jordan and Einstein frames to the Brans-Dicke gravity, and for larger ω, the similar results were proposed as those in general relativity. The similar problem was studied in [62], by considering Bianchi identity as a consistency condition. For the IR cut-off as a future event horizon, the importance of Brans-Dicke gravity for the dust matter and the HDE has been explored in [63]. It is proposed that with the Hubble radius as infrared cut-off, the standard HDE may not produce the accelerated expansion Universe in the Brans-Dicke gravity, but the suitable description of the Universe can be obtained by taking the IR cut-off as a future event horizon [64]. Therefore, many other works proposed that the Brans-Dicke gravity is suitable for the examination in the holographic dark energy scenario [65–71]. Observational constraints also have been proposed for a sign-changeable interaction among the Universe sectors [72–74]. Considering different IR cut-offs, the noninteracting and interacting Tsallis HDE and their cosmological consequences are explored in the Brans-Dicke theory [75–77].

Very recently, the authors constructed the noninteracting RHDE model in the Brans-Dicke theory taking the Hubble horizon as the IR cut-off [78]. While in this work, we propose the interacting RHDE model in the framework of the Brans-Dicke theory in both flat and nonflat Universes. The paper is organized as follows: we explored the interacting RHDE model and physical parameters in the Brans-Dicke theory in Section 2. We study the stability of the RHDE model in Section 3. The conclusion is given in the last section.

Throughout the text, an "overdot" represents a derivative with respect to cosmic time.

2. Interacting Rényi Holographic Dark Energy in the Brans-Dicke Theory

We consider a homogeneous and isotropic FLRW Universe which is described by the line element

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \quad (1)$$

where $a(t)$ is the scale factor of the Universe, t is the cosmic time, and the curvature constant $k = +1, 0, -1$ corresponds to closed, flat, and open Universes, respectively. The coordinates r, θ, and ϕ are known as *comoving* coordinates.

In BD theory, the action is given by [57, 79]

$$S = \frac{1}{16\pi G}\int \sqrt{-g}\left[\phi R - \omega \frac{\phi_{,\alpha}\phi^{,\alpha}}{\phi} + L_m\right] d^4x, \quad (2)$$

where ϕ is the BD scalar field, R is the Ricci scalar, ω is the BD parameter, and L_m is the Lagrangian matter. Here, the gravitational constant (G) takes the place of the time-dependent scalar field ϕ, which is inversely proportional to G, i.e., $\phi(t) = 1/8\pi G$. If we assume the matter field to consist of a perfect fluid, then the BD field equations from the variation of action (2) and for the FLRW space-time are obtained as [79]

$$\frac{3}{4w}\phi^2\left(\frac{k}{a^2} + H^2\right) + \frac{3H\phi\dot{\phi}}{2w} - \frac{\dot{\phi}^2}{2} = \rho_D + \rho_m, \quad (3)$$

$$\frac{-\phi^2}{4w}\left(\frac{k}{a^2} + \frac{2\ddot{a}}{a} + H^2\right) - \frac{H\phi\dot{\phi}}{w} - \frac{\phi\ddot{\phi}}{2w} - \frac{1}{2}\left(\frac{1}{w} + 1\right)\dot{\phi}^2 = p_D, \quad (4)$$

where $H = \dot{a}/a$ is the Hubble parameter, ρ_m is the matter energy density, ρ_D is the RHDE density, and p_D is the RHDE pressure. The BD scalar field evolution equation is

$$\ddot{\phi} + 3H\dot{\phi} - \phi\frac{3}{2w}\left(H^2 + \frac{k}{a^2} + \frac{\ddot{a}}{a}\right) = 0. \tag{5}$$

2.1. Rényi Entropy and HDE. It is important to mention here that it seems there is a deep connection between quantum gravity and generalized entropy scenarios, and indeed, quantum aspects of gravity may also be considered as another motivation for considering generalized entropies [22, 80]. Tsallis entropy is one of the generalized entropy measures which lead to acceptable results in the gravitational and different cosmological setups [23, 25, 43, 81–89]. Usually, Tsallis entropy is defined as [86]

$$S_{\mathrm{TE}} = \frac{1}{1-N}\sum_{i=1}^{W}\left(P_i^N - P_i\right), \tag{6}$$

for a system consisting of W discrete states. In the above equation, P_i is the ordinary probability of accessing state i, and N is a real parameter which may be originated from the nonextensive features of the system such as the long-range nature of gravity [22, 83, 86]. In fact, the concept of nonextensivity is more complex than that of nonadditivity [87]. For example, the well-known Bekenstein entropy is nonadditive and nonextensive simultaneously (for details, see Refs. [84, 85]). It is proposed recently that the Bekenstein entropy ($S = A/4$, where $A = 4\pi L^2$ and L is the IR cut-off) is actually a Tsallis entropy leading to

$$S = \frac{1}{\delta}\log\left(\frac{\delta}{4}A + 1\right) = \frac{1}{\delta}\log\left(\pi\delta L^2 + 1\right), \tag{7}$$

for the Rényi entropy content of the system [21, 22]. Here, δ is a free parameter and known in the current literature as the real nonextensive parameter that quantifies the degree of nonextensibility [22, 86, 87]. It is proposed in [90] that the δ parameter affects the energy balance of the Universe. When $\delta < 1$, the gravitational field is strong enough in such a way that we need only a small quantity of DE and DM to construct the observable Universe. On the other hand, when $\delta > 1$, the gravitational field is weak in such a way that we need, contrarily to the $\delta < 1$ case, a larger quantity of DE and DM. To sum up, $\delta < 1$ implies less DE and $\delta > 1$ implies more DE than we would have if we consider the standard Boltzmann-Gibbs scenario [90, 91]. In [22], the authors used the value of δ from −1400 to −900. There are wide ranges for δ which can produce desired results, while we have taken the values of δ from −1600 to −1400. Authors investigated late-time acceleration for a spatially flat dust filled Universe in the Brans-Dicke theory in the presence of a positive cosmological constant Λ, where the value for the Brans-Dicke-coupling constant w is taken as 40,000 [92]. Authors have studied the Tsallis holographic dark energy in the Brans-Dicke framework using $b^2 = 0.05, 0.10, 0.15$ and $n = 0.001, 0.005, 0.05$ [77]. The primary focus is in [93] on the FLRW Universe specified by WMAP data. The role of dark energy is played by the vacuum energy density in this model, that is, one had $\Lambda^4 \sim \rho_\Lambda \equiv \rho_D$. With the assumption $\rho_d \propto TdS$ [22] and $L = 1/H$ (i.e., Hubble horizon) and using Equation (7), we obtained energy density for the RHDE as

$$\rho_D = \frac{3c^2H^2}{\left(\pi\delta/H^2 + 1\right)8\pi}, \tag{8}$$

where c^2 is a numeric constant. We used $T = H/2\pi$ and $A = 4\pi/H = 4\pi(3V/4\pi)^{2/3}$; relations to get this equation corroborated in a flat FLRW Universe [94]. One can get $\rho_D = 3c^2H^2/8\pi$ without δ, which is in complete agreement with the standard HDE [14–18]. It deserves mention here that the apparent horizon is a proper causal boundary for the cosmos in agreement with the thermodynamics laws. Besides, in a flat FLRW Universe, Friedmann equations indicate that whenever DE is dominant in the cosmos, its energy density will scale with H^2 (for details, see [22, 94]). Therefore, from a thermodynamic point of view, a HDE model in the flat FLRW Universe, for which the radii of the apparent horizon and the Hubble horizon $(1/H)$ are the same, will be more compatible with the thermodynamics laws, if it can provide a proper description for the Universe by using the Hubble horizon as its IR cut-off. Following [68], we assume that $\phi \propto a^n$, i.e., the power law of scale factor in this case to the BD scalar field ϕ. One can now easily obtain

$$\dot{\phi} = n\phi\frac{\dot{a}}{a}, \tag{9}$$

and hence,

$$\ddot{\phi} = H^2n^2\phi + \phi n\dot{H}. \tag{10}$$

The Rényi HDE density with the Hubble horizon as the IR cut-off is given as

$$\rho_D = \frac{3c^2H^2\phi^{2\delta}}{8\pi\left(\pi\delta/H^2 + 1\right)}. \tag{11}$$

Here, the holographic principle [17] is used, and the effective gravitational constant G_{eff} is given by $G_{\mathrm{eff}} = w/2\pi\phi^2$. The gravitational constant G may be found from G_{eff} as a limit. The RHDE energy density can be recovered in the fundamental cosmology [22]. The Holographic DE can also be found in the Brans-Dicke gravity for the case $\delta = 1$ [64]. The dimensionless density parameters are defined as

$$\Omega_m = \frac{4w\rho_m}{3\phi^2H^2}, \Omega_D = \frac{c^2H^2w\phi^{2\delta-2}}{2\pi\left(\pi\delta + H^2\right)}, \Omega_k = \frac{k}{a^2H^2}, \Omega_\phi = 2n\left(\frac{nw}{3} - 1\right). \tag{12}$$

Our main goal of this work is to build a cosmological model of late acceleration based on the BD theory of gravity and on the assumption that the RHDE and the pressureless dark matter do not conserve separately. Therefore, we assume that both components—the RHDE and the pressureless matter—interact with each other, i.e., one component may grow at the expense of the other. Hence, the energy conservation equations for them are given as follows:

$$\dot{\rho}_D + 3\rho_D(1+\omega_D) = -Q, \tag{13}$$

and

$$\dot{\rho}_m + 3H\rho_m = Q, \tag{14}$$

where $\omega_D = p_D/\rho_D$ represents the Rényi HDE equation of state (EoS) parameter and Q denotes the interaction term. Clearly, for $Q<0$ ($Q>0$), there is an energy flow from pressureless matter (RHDE) to RHDE (pressureless matter). We assume the form of interaction as$Q = 3b^2 Hq(\rho_D + \rho_m)$ [41, 42, 74], in which b^2 is the coupling constant and q denotes the deceleration parameter. Here, the main ingredient is the deceleration parameter q ($\equiv -\ddot{a}/aH^2$) in the interaction term Q, and hence, Q can change its sign when the expansion of our Universe changes from the early decelerated ($q>0$) phase to the late-time accelerated ($q<0$) phase. So, the above interacting term deserves further investigation in the present context. Now, taking derivatives with respect to time of Equation (11), we get

$$\dot{\rho}_D = 2H\rho_D\left(\delta n + \left(\frac{\pi\delta}{\pi\delta + H^2} + 1\right)\frac{\dot{H}}{H^2}\right), \tag{15}$$

combined with relation $\Omega_D{}' = \dot{\Omega}_D/H$ to obtain

$$\Omega_D{}' = 2\Omega_D\left(n(\delta - 1) + \left(\frac{\pi\delta}{\pi\delta + H^2}\right)\frac{\dot{H}}{H^2}\right), \tag{16}$$

where the prime denotes derivative with $x = \log a$. Now, taking the derivative with respect to time of Equation (3) and substituting the value of $\dot{\phi}$, $\ddot{\phi}$, $\dot{\rho}_m$, and $\dot{\rho}_D$ from Equations (9), (10), (14), and (15), respectively, we get

$$\frac{\dot{H}}{H^2} = -\frac{(\pi\delta + H^2)\left(3\left(3b^2 - 2n + 5\right)\Omega_k - 9b^2 + \Omega_D(6\delta n + 9) + 2n(2n(nw-3)-3) - 9\right)}{H^2\left(-9b^2 + 6\Omega_D + 4n(nw-3) - 6\right) + \pi\delta\left(-9b^2 + 12\Omega_D + 4n(nw-3) - 6\right) + 9b^2\left(\pi\delta + H^2\right)\Omega_k}. \tag{17}$$

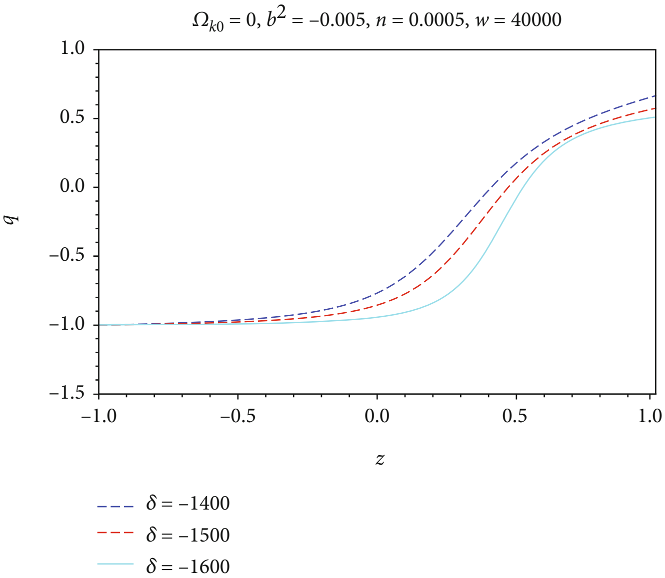
Defining, as usual, the deceleration parameter as

$$q = -\frac{\ddot{a}}{aH^2} = -1 - \frac{\dot{H}}{H^2}, \tag{18}$$

and using Equation (17), we obtain

$$q = -1 + \frac{(\pi\delta + H^2)\left(3\left(3b^2 - 2n + 5\right)\Omega_k - 9b^2 + \Omega_D(6\delta n + 9) + 2n(2n(nw-3)-3) - 9\right)}{H^2\left(-9b^2 + 6\Omega_D + 4n(nw-3) - 6\right) + \pi\delta\left(-9b^2 + 12\Omega_D + 4n(nw-3) - 6\right) + 9b^2\left(\pi\delta + H^2\right)\Omega_k}. \tag{19}$$

The evolutionary behavior of the deceleration parameter is plotted for the interacting Rényi HDE model versus redshift z by finding its numerical solution using the initial values $\Omega_{D0} = 0.70$ and $H_0 = 72.30$, for both flat Universe $\Omega_k = 0$ and nonflat Universe $\Omega_k = 0.012$. It is proposed by different observations that the Universe is in an accelerated expansion phase, and the value of the deceleration parameter lies in the range $-1 \le q < 0$. Also, we have used $n = 0.0005$ [77] for all plots. All the physical parameters are examined through δ and coupling coefficient b^2 because they play a crucial role in the evolution of dynamical parameters of the RHDE. From Figure 1, we see the evolutionary behavior of q for interacting RHDE in BD gravity, for distinct estimations of b^2 and δ in the nonflat Universe (lower two panels) and flat Universe (upper two panels). We can observe from Figure 1 that the RHDE model shows the transition from an early decelerated stage to a current accelerated stage for both cases for distinct estimations of b^2 and parameter δ. In this context, it is worthwhile mentioning that the standard HDE in the framework of BD theory can explain the accelerated expansion if the event horizon is taken as the role of the IR cut-off [64]. Such a scenario also predicts no acceleration if the Hubble horizon is considered as the IR cut-off. Therefore, the novelty of the present work is that it can explain the current accelerated phase of the Universe if we choose the IR cut-off to be the Hubble horizon.



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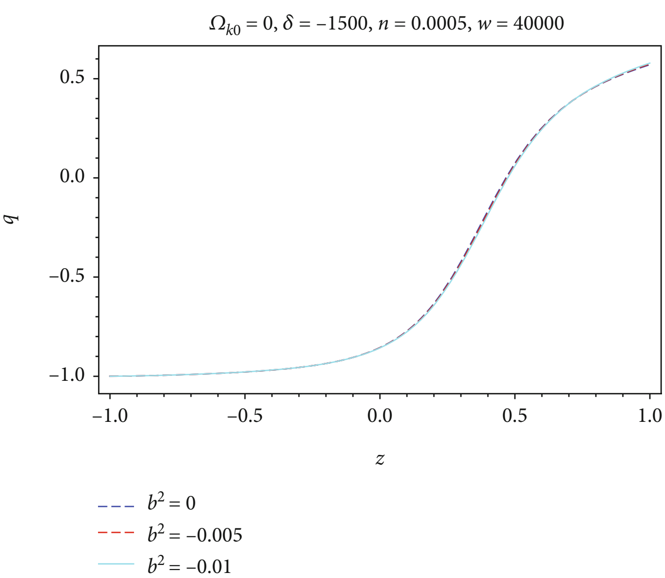

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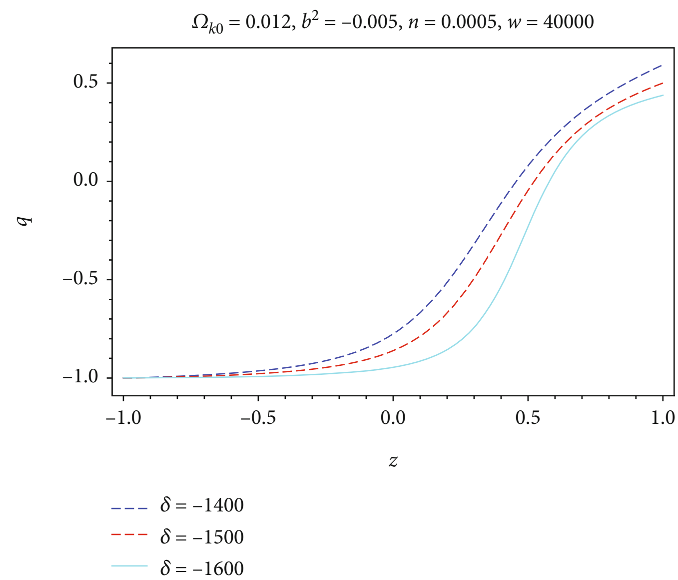
FIGURE 1: Continued.



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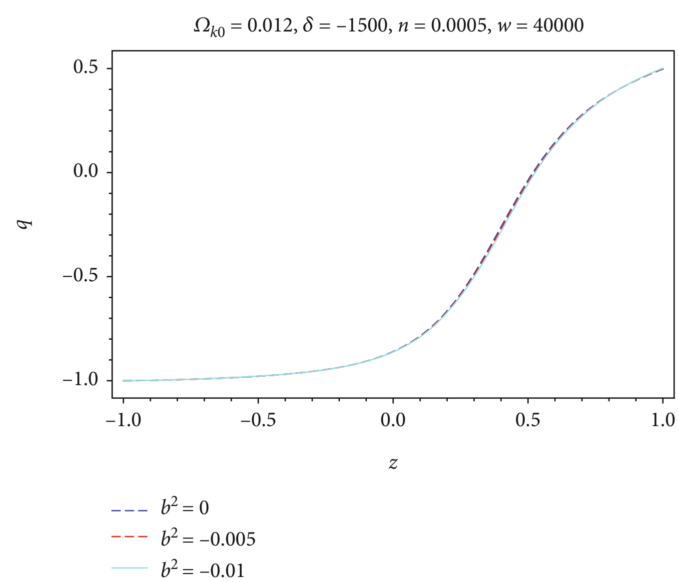


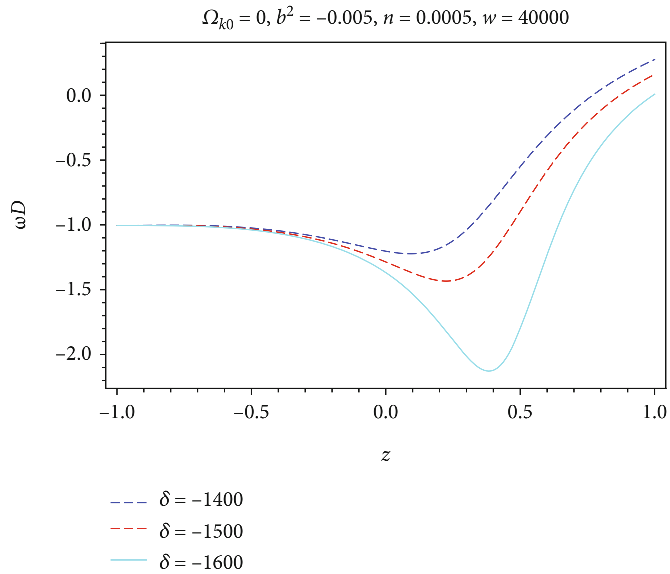
(d)

Figure 1: The deceleration parameter (q) evolutionary behavior versus redshift for nonflat Universe (c, d) and flat Universe (a, b) for distinct values of δ and b^2. Here, $H_0 = 72.30$ and $\Omega_{D0} = 0.704$.

Combining Equations (13), (15), and (17) with each other, the EoS parameter is obtained as

$$\begin{aligned}\omega_D = {}& \left[3\Omega_D\left(H^2\left(-9b^2 + 6\Omega_D + 4n(nw-3) - 6\right)\right.\right.\\ &\left.\left. + \pi\delta\left(-9b^2 + 12\Omega_D + 4n(nw-3) - 6\right) + 9b^2\left(\pi\delta + H^2\right)\Omega_k\right)\right]^{-1}\\ &\times\left[\left(H^2\left(2\Omega_D\left(-4(\delta-1)n^3 w - 6n^2(-2\delta + w + 2)\right.\right.\right.\right.\\ &\left. + 6(\delta+2)n + 3(5-2n)\Omega_k\right) - 3b^2(\Omega_k - 1)(3(2n-5)\Omega_k\\ &\left. + 2n(2n(-nw + w + 3) - 3) + 3)\right) + \pi\delta\left(-3b^2(\Omega_k - 1)(3(2n-5)\Omega_k\right.\\ &+ 2n(2n(-nw + w + 3) - 3) + 3) - 2\Omega_D\left(6(2n-5)\Omega_k + 4(\delta-2)n^3 w\right.\\ &\left.\left.\left.\left. + 6n^2(-2\delta + w + 4) - 6(\delta+1)n + 9\right)\right)\right)\right].\end{aligned} \tag{20}$$

We have graphed the behavior of EoS parameter ω_D of our derived interacting RHDE model for both $\Omega_k = 0$ (two upper panels) and $\Omega_k = 0.012$ (two lower panels) cases, in Figure 2 for distinct values of parameter δ and coupling coefficient b^2. According to this figure, it can be seen that ω_D of the RHDE model varies from quintessence to the phantom region ($\omega_D < -1$). Moreover, we can observe that the EoS parameter approaches ΛCDM model ($\omega_D = -1$) for all values of δ and b^2 in the future, which is in agreement with cosmological observations. We also noted that the evolution of the EoS parameter at an early time in both flat and nonflat Universes is more distinct for different values of δ in comparison to coupling coefficient b^2.



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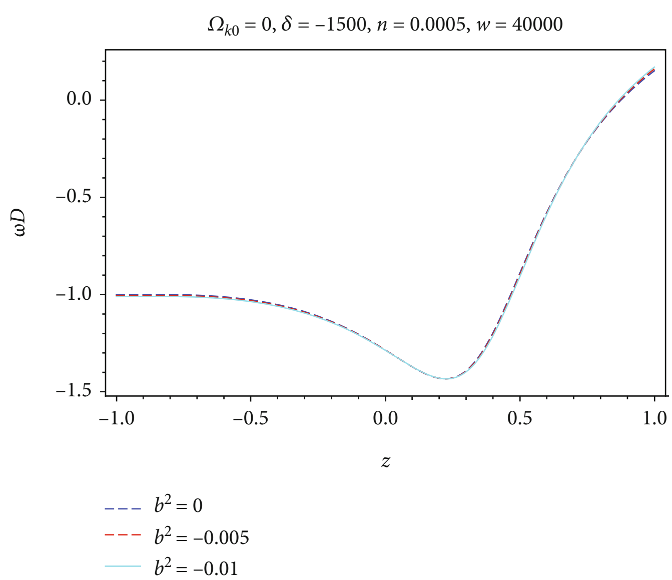

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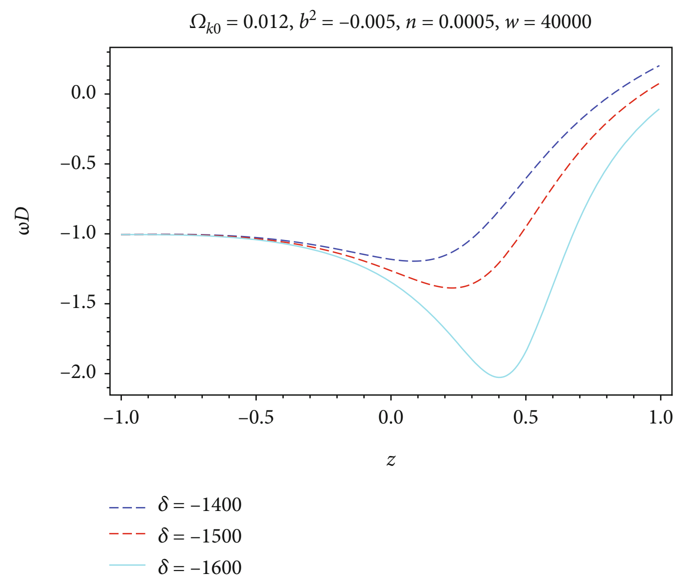

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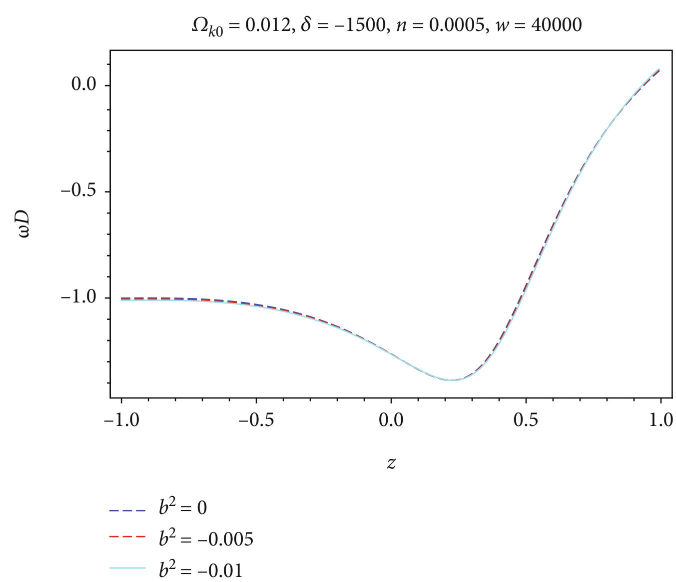


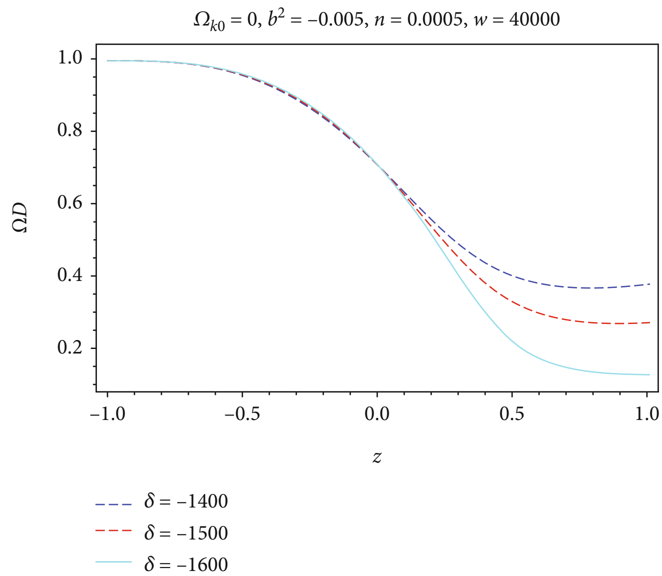
(d)

FIGURE 2: Evolutionary behavior of the EoS parameter ω_D against redshift for interacting RHDE for nonflat Universe (c, d) and flat Universe (a, b) for distinct values of δ and b^2. Here, $\Omega_{D0} = 0.704$ and $H_0 = 72.30$.

By putting Equation (17) in Equation (16), we also obtain the evolution of dimensionless RHDE density parameter as

$$\Omega_D{}' = \left[(\delta - 1)n + 2\Omega_D \times \frac{\pi\delta\left(-3\left(3b^2 - 2n + 5\right)\Omega_k + 9b^2 - 3\Omega_D(2\delta n + 3) + 2n\left(-2n^2 w + 6n + 3\right) + 9\right)}{H^2\left(-9b^2 + 6\Omega_D + 4n(nw - 3) - 6\right) + \pi\delta\left(-9b^2 + 12\Omega_D + 4n(nw - 3) - 6\right) + 9b^2\left(\pi\delta + H^2\right)\Omega_k}\right]. \tag{21}$$

We have shown the behavior of interacting RHDE density parameter Ω_D in Figure 3 for both $\Omega_k = 0$ (two upper panels) and $\Omega_k = 0.012$, (two lower panels) cases, for distinct values of coupling coefficient b^2 and δ. The thermal history of



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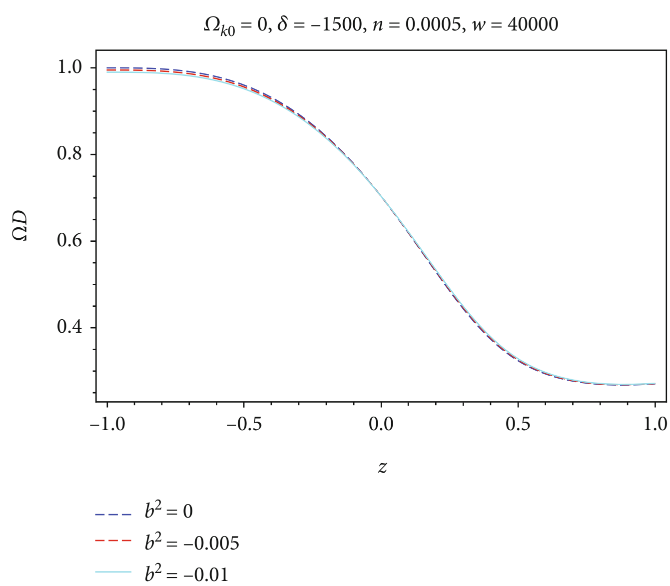

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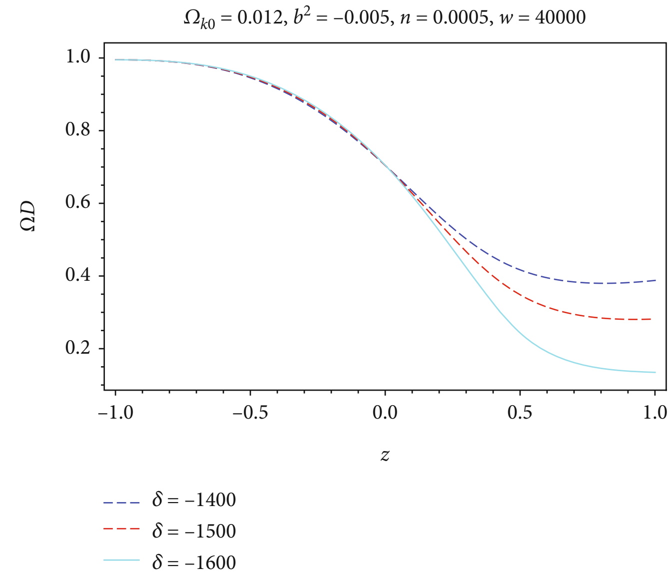

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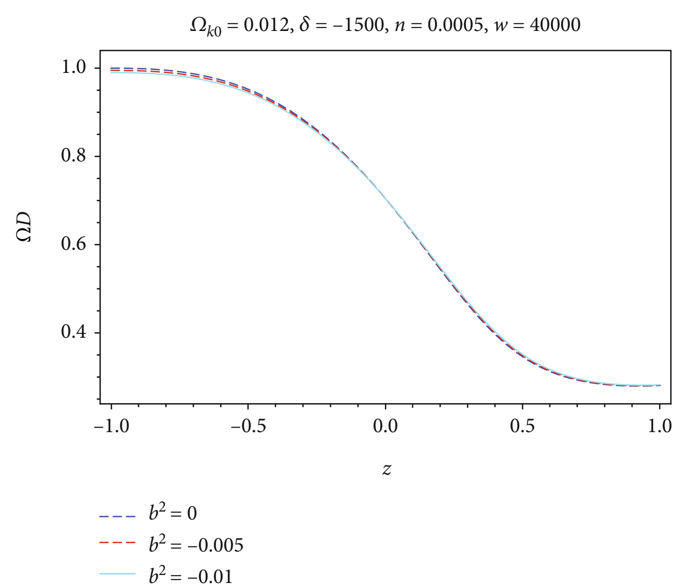


(d)

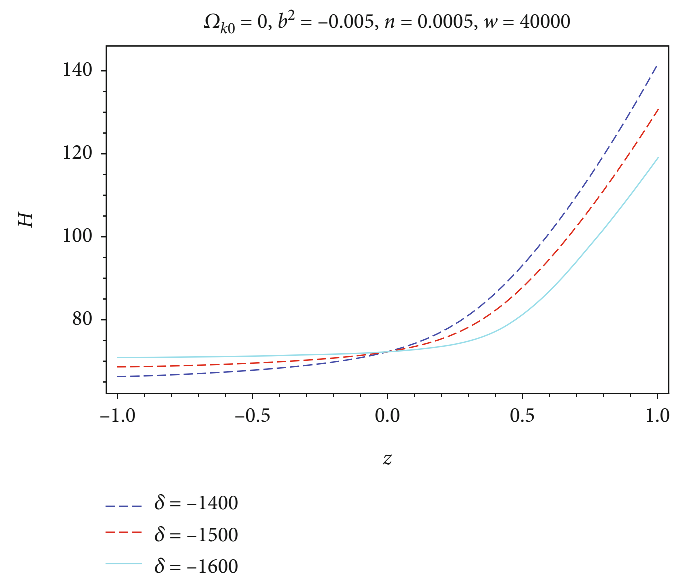
Figure 3: The evolutionary behavior of Ω_D against z for interacting RHDE for nonflat Universe (c, d) and flat Universe (a, b) for different values of δ and b^2. Here, $\Omega_{D0} = 0.704$ and $H_0 = 72.30$.

the Universe, in particular, the successive sequence of matter and DE era, can be observed from these figures for different values of δ and b^2 in both nonflat and flat Universes. We also observed that the RHDE density parameter is consistent with cosmological observations [7], and our results are consistent.

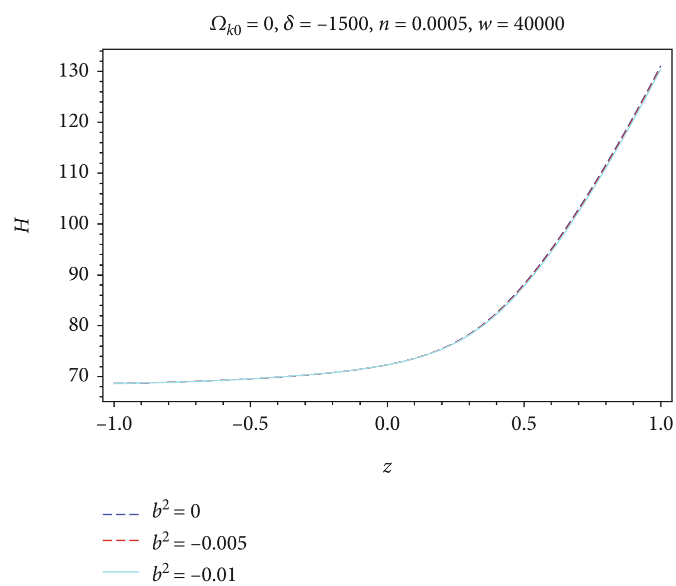
We have plotted the behavior of the Hubble parameter H of our derived interacting RHDE model for both the $\Omega_k = 0$ (two upper panels) and $\Omega_k = 0.012$, (two lower panels) cases, in Figure 4 for distinct values of parameter δ and coupling coefficient b^2. It depicts that the variation of δ affects the behavior of H, while different values of coupling coefficient b^2 do not affect it. The value of H decreases and approaches to a positive value near 70 in the far future.

3. Stability

In this section, we shall discuss the stability of the interacting RHDE model through the squared sound speed v_s^2 in both flat and nonflat Universes. The $v_s^2 \geq 0$ (the real value of speed), shows a regular propagating mode for a density perturbation. For $v_s^2 < 0$, the perturbation becomes an irregular wave equation. Hence the negative squared speed (imaginary value of speed) shows an exponentially growing mode for a density perturbation. That is, an increasing density

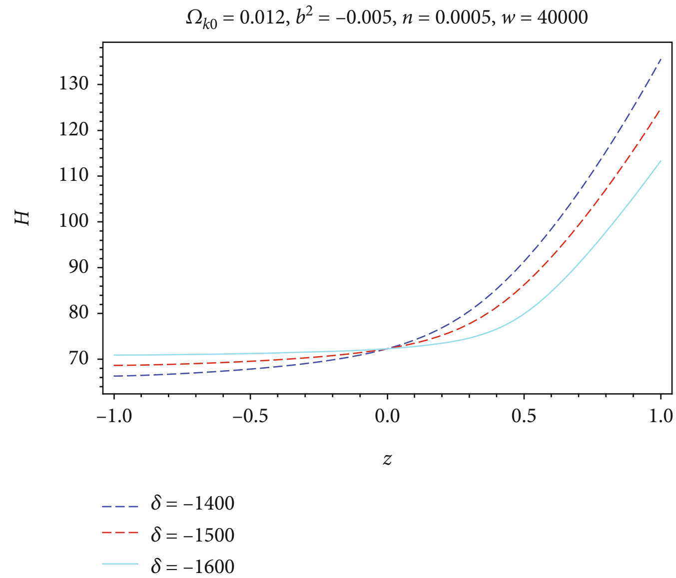


(a)

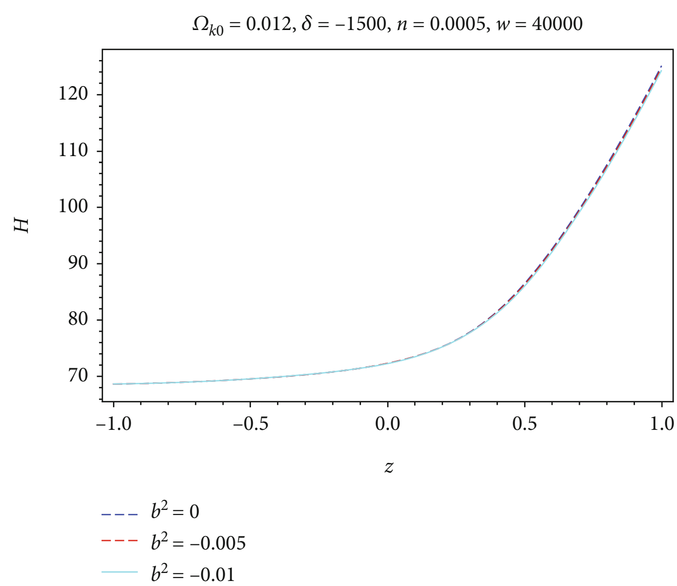


(b)

Figure 4: Continued.



(c)



(d)

Figure 4: The Hubble parameter (H) evolutionary behavior versus redshift for nonflat Universe (c, d) and flat Universe (a, b) for distinct values of δ and b^2. Here, $H_0 = 72.30$ and $\Omega_{D0} = 0.704$.

perturbation induces a lowering pressure, supporting the emergence of instability [95].

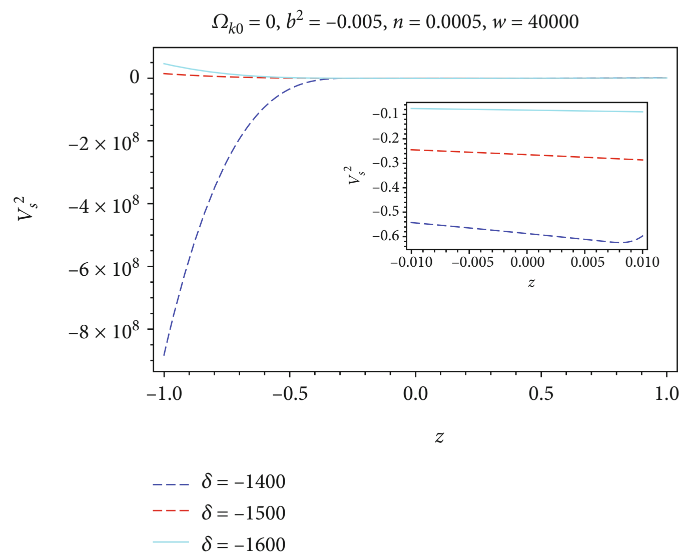
The squared sound speed is given as [96, 97]

$$v_s^2 = \frac{dp_D}{d\rho_D} = \frac{\rho_D \dot{\omega}_D}{\dot{\rho}_D} + \omega_D. \quad (22)$$

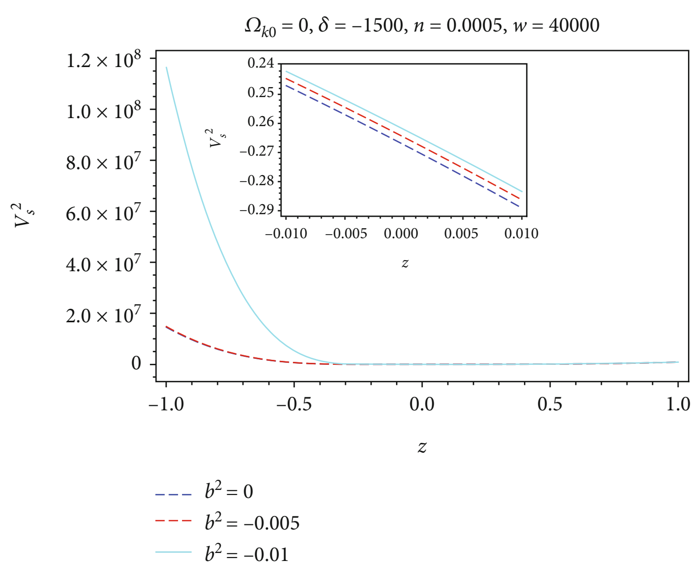
Now, inserting the result of Equation (15) in Equation (22), we get

$$v_s^2 = \omega_D + \frac{\dot{\omega}_D}{2H\left(\left(\pi\delta/\left(\pi\delta + H^2\right) + 1\right)\left(\dot{H}/H^2\right) + \delta n\right)}. \quad (23)$$

We solve the Equation (23) numerically by using Mathematica package NDSolve and plotted the squared sound speed v_s^2 versus redshift z in Figure 5, for both $\Omega_k = 0$ (two upper panels) and $\Omega_k = 0.012$, (two lower panels) cases for distinct values of the parameter δ and coupling coefficient b^2.From Figure 5, we observe that the RHDE model is not stable initially by taking different values of δ and b^2 in both flat and nonflat Universes, while for $\delta = -1400$ in both flat and nonflat Universes, the value of the squared sound speed v_s^2 diverges. By taking different values of b^2 in both flat and nonflat Universes, the RHDE model becomes stable at the late time. By analyzing all these plots, we can say that values of δ and b^2 have qualitative effects on the nature of the

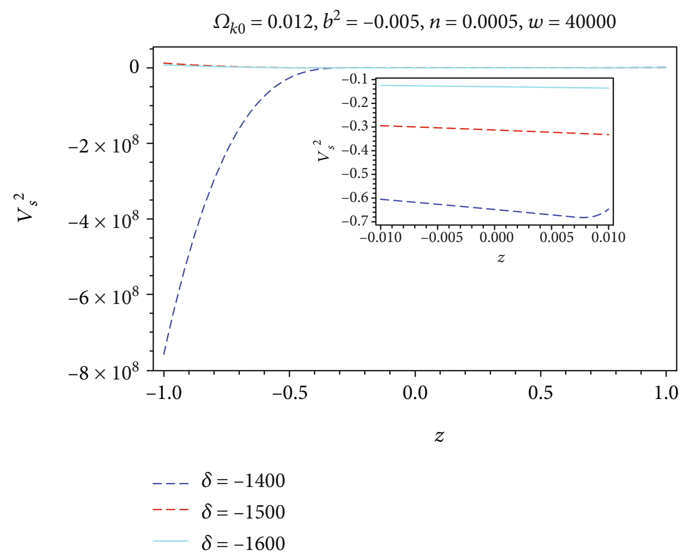


(a)

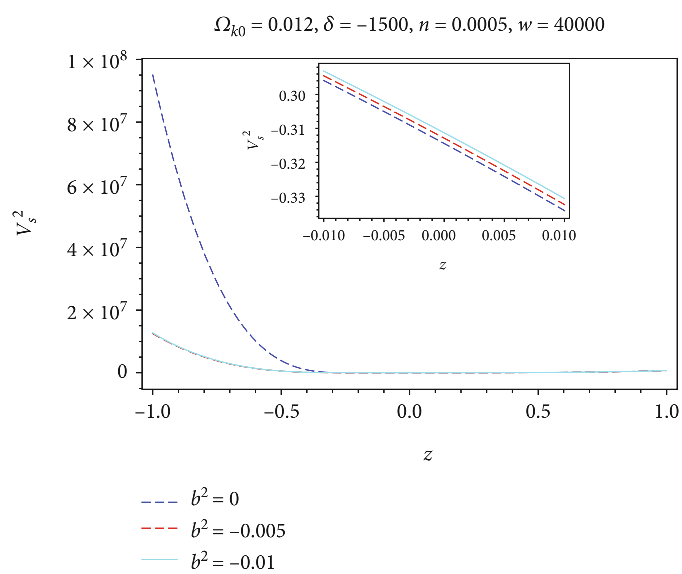


(b)

Figure 5: Continued.



(c)



(d)

FIGURE 5: The behavior of v_s^2 against z for the interacting Rényi HDE for nonflat Universe (c, d) and flat Universe (a, b) for distinct values of δ and b^2. Here, $\Omega_{D0} = 0.704$ and $H_0 = 72.30$.

squared sound speed v_s^2 in both the nonflat and the flat cosmos. The inset plot of Figure 5 shows a close-up of the outer plot around $z = 0$ in which the difference can be seen. They are not exactly identical but the difference is very small.

4. Concluding Remarks

In this work, we explored the role of the interacting FLRW cosmos to model dark energy in the Brans-Dicke theory framework using RHDE by taking an infrared cut-off as the Hubble radius in both nonflat and flat Universes. The pressureless matter is assumed to interact with the RHDE through a sign-changeable interaction. In this analysis, we have used the initial values $\Omega_{D0} = 0.70$, $\Omega_{m0} = 0.30$, $H_0 = 72.30$, and $n = 0.005$ [77] for both flat Universe ($\Omega_k = 0$) and nonflat Universe ($\Omega_k = 0.012$). It has been found that for different values of the Rényi parameter δ and the coupling coefficient b^2, the interacting RHDE model produces the suitable behavior for the deceleration parameter (q), the EoS parameter (ω_D), the RHDE density parameter (Ω_D), and the Hubble parameter, in both the cases (see Figures 1–4). The effect of different values of δ and b^2 is only quantitative on these parameters.

As discussed earlier, the Brans-Dicke theory in the framework of HDE can explain the accelerated expansion if we choose the IR cut-off to be the event horizon. The theory also predicts no acceleration if we choose the IR cut-off to be the Hubble horizon. However, in our case, the deceleration parameter q shows a smooth transition from the decelerated phase ($q > 0$) early to the accelerated phase ($q < 0$) at a later time. Hence, a remarkable feature of this model is that RHDE in the framework of the Brans-Dicke theory explains the accelerated expansion if we choose the IR cut-off to be the Hubble horizon. It has also been found that the EoS parameter ω_D varies from a quintessence ($\omega_D > -1$) to the phantom region ($\omega_D < -1$), and the RHDE model transits decelerating to an accelerating stage of the Universe and ω_D approach to -1 as $z \rightarrow -1$, which implies that the RHDE model imitates the cosmological constant at a far future. It is observed that the RHDE density parameter Ω_D becomes 1 as $z \rightarrow -1$. Moreover, it is observed that the variation of δ affects the behavior of the Hubble parameter H, while different values of b^2 do not affect it. Also, the value of H decreases and approaches to a value near 70 in the far future. Furthermore, we have investigated the classical stability of our model by analyzing the squared sound speed v_s^2. It has been found that the stability of our model crucially depends on the choices of the parameter δ in both flat and nonflat Universes (see Figure 5).

As we showed, the present model exhibits more interesting phenomenology comparing to the standard scenario, and hence, it can be a candidate for the description of nature. In a follow-up study, we would like to perform an observational analysis to constrain the parameter δ.

Data Availability

This manuscript has no associated data or the data will not be deposited. (Authors' comment: data sharing is not applicable to this article as no new data were created or analyzed in this study).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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