

## Research Article

# The New Scramble for Faure Sequence Based on Irrational Numbers

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Received 24 December 2020; Revised 11 January 2021; Accepted 10 March 2021; Published 28 April 2021

Academic Editor: Leopoldo Greco

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This article intends to review quasirandom sequences, especially the Faure sequence to introduce a new version of scrambled of this sequence based on irrational numbers, as follows to prove the success of this version of the random number sequence generator and use it in future calculations. We introduce this scramble of the Faure sequence and show the performance of this sequence in employed numerical codes to obtain successful test integrals. Here, we define a scrambling matrix so that its elements are irrational numbers. In addition, a new form of radical inverse function has been defined, which by combining it with our new matrix, we will have a sequence that not only has a better close uniform distribution than the previous sequences but also is a more accurate and efficient tool in estimating test integrals.

## 1. Introduction

It is well known that Monte Carlo calculations are based on the generation of random numbers on interval  $(0,1)$ . Therefore, the generation of random numbers that have more uniformity on  $(0,1)$  guarantees better approximations in these calculations. In recent years, some researchers have employed quasirandom sequences instead of random numbers to aim producing extra uniformity of the randomly generated numbers on  $(0,1)$ . Due to the breadth and complexity of some problems that are mostly unsolvable by classical mathematical methods or solving them with classical methods is associated with more time and computational cost, the stochastic solving of such cases with numerical methods and using the Monte Carlo method plays a key role. The quasirandom sequences are common in Monte Carlo calculations such as Faure, Halton, Niederreiter, and Sobol sequences, but due to the lack of complete success of these sequences in Monte Carlo computation, we use scrambled versions of them, all of which are designed to increase the uniformity of randomly (quasirandom) generated numbers

on  $(0,1)$ , so that we can estimate the obtained solution to the desired unknown solution of the problem.

To resolve this problem, researchers are competing on the use of scrambled quasirandom generators based on their version of random number generation to provide more accurate results in Monte Carlo calculations.

Today, Monte Carlo and quasi-Monte Carlo methods are widely used to solve the computations of physical and mathematical problems. Quasi-Monte Carlo (QMC) methods play an alternative role for Monte Carlo methods. The advantage of these methods is that they use numbers to provide extra uniformity on unit hypercube. This feature has led to the use of these methods to estimate high-dimensional integrals (Niederreiter, 1992; Spanier and Maize, 1994) [1].

So far, several quasirandom sequences (or low discrepancy sequences) have been introduced for the QMC method. Such as the Faure sequence, the Halton sequence, and the Sobol sequence. Despite the fact that among these three sequences, the Faure sequence has better features in terms of discrepancy bound, but in practice, it is less used. Because, the convergence rate of this class of sequences is not so good

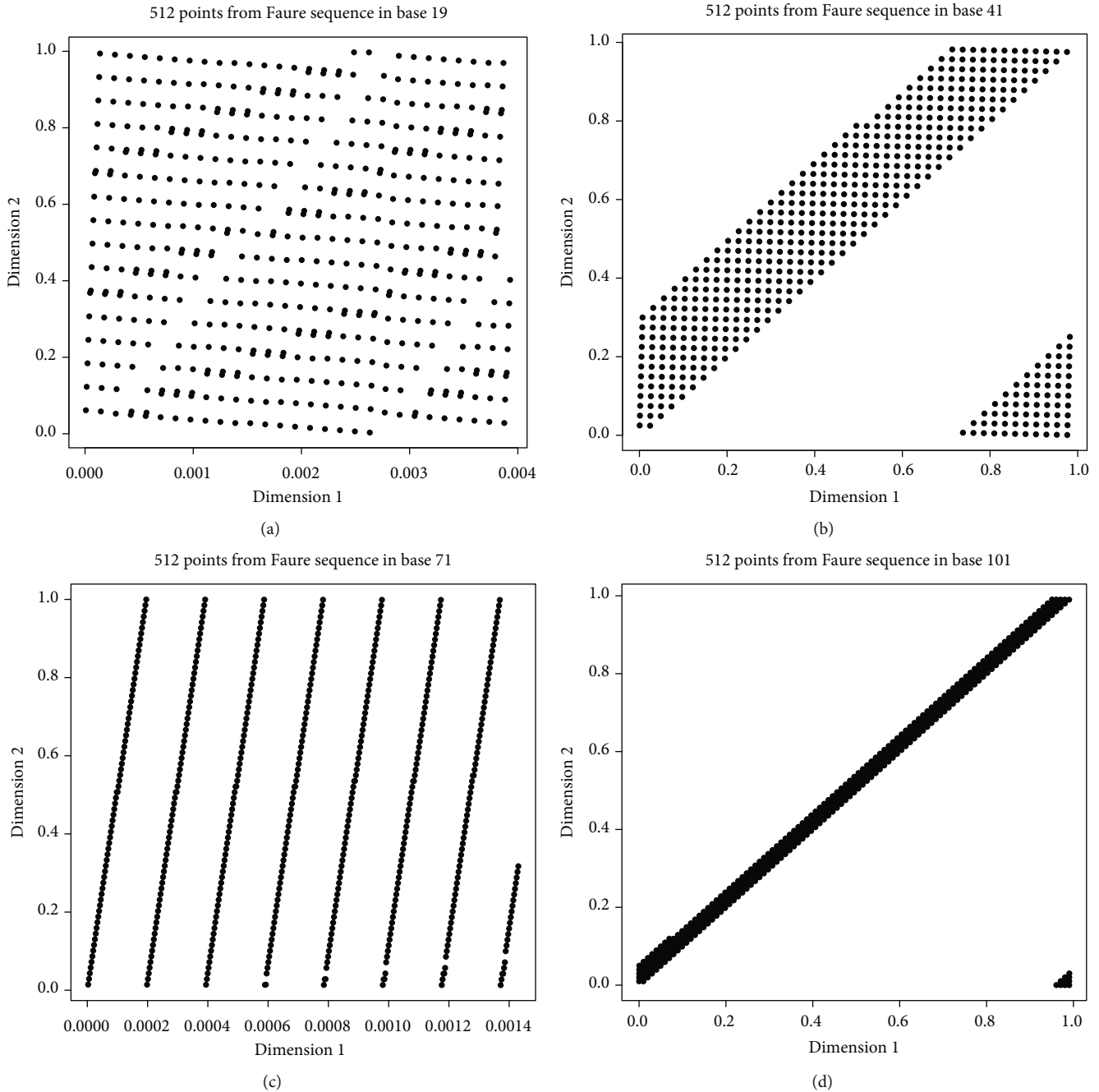


FIGURE 1: 512 points from the original Faure sequence in several bases.

compared with the other sequences [2]. In addition, due to the correlation between the different dimensions of the Faure sequence, the distribution of the sequence points is not very favorable, and we see poor two-dimensional projections (Figure 1). To overcome this problem, many scrambling methods have been proposed for the Faure sequence (see [3]). In almost all of these scrambles, there are attempts to define a new matrix by shifting numbers in the generating matrix or the placement of the elements, and there has been less talk about the properties of irrational numbers. The matrix that we introduce in this paper has been selected from

several proposed matrices. Because it has good two-dimensional projections and it is also at a very high level in terms of integral estimation.

In the next section, the structure of the original Faure sequence is given. We then briefly list the scramblers that have already been introduced in Section 3. In Section 4, we have brought our proposed matrix. Sections 5 and 6 give the evaluation criteria for the quality of the sequence generated by our proposed matrix and compare it with previous sequences, and in Section 7, the conclusion is stated.

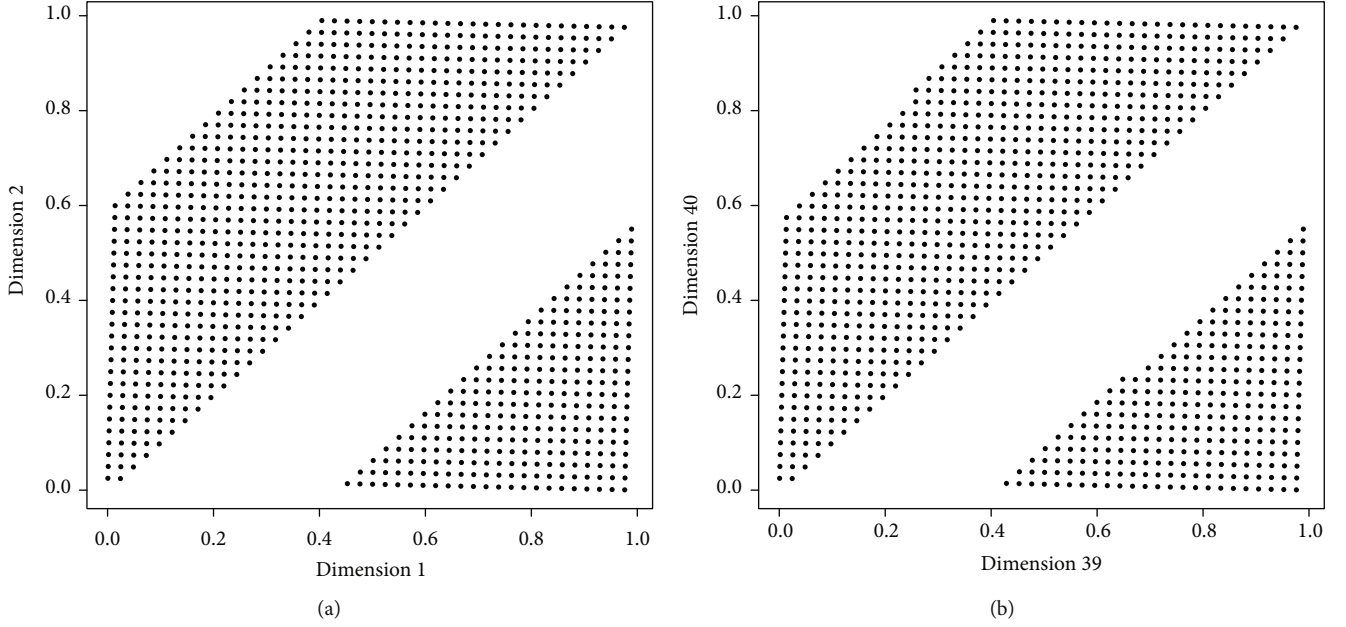


FIGURE 2: 1000 points from the original Faure sequence.

## 2. The Faure Sequence

Suppose  $p \geq 2$  is a prime number, and suppose  $N$  is the number of points we want to generate, and  $\mathbf{n} = (a_0, a_1, \dots, a_{m-1})^T$  is a vector of integers whose components are from the expansion of the number  $n$  on the base  $p$ , where  $0 \leq a_j \leq p$  and  $m = \lceil \log_p N \rceil$ . We define the radical inverse function,  $\phi_p(\mathbf{n})$ , as

$$\phi_p(\mathbf{n}) = \frac{a_0}{p} + \frac{a_1}{p^2} + \dots + \frac{a_{m-1}}{p^m}. \quad (1)$$

For the Faure sequence, we define a different generator matrix for each dimension. If  $\mathbf{P}$  be the *Pascal matrix*, then for a  $s$ -dimensional Faure sequence the generator matrix of the  $j$ th dimension is  $\mathbf{C}^{(j)} = \mathbf{P}^{j-1}$ ,  $1 \leq j \leq s$ , where the member on the row  $c$  and the column  $r$  is defined as follows:

$$\mathbf{P}^{j-1} = \binom{c-1}{r-1} (j-1)^{c-r} \pmod{p}, \quad c \geq 1, r \geq 1. \quad (2)$$

Thus, let  $x_n^{(j)}$  be represents the number  $n$  in the dimension  $j$  in the Faure sequence, then

$$x_n^{(j)} = \phi_p(\mathbf{P}^{j-1} \mathbf{n}), \quad (3)$$

and so the  $s$ -dimensional Faure sequence is  $(\phi_p(\mathbf{P}^0 \mathbf{n}), \phi_p(\mathbf{P}^1 \mathbf{n}), \dots, \phi_p(\mathbf{P}^{s-1} \mathbf{n}))$ .

## 3. Scrambling the Faure Sequence

Since the introduction of the Faure sequence, several methods were proposed to scramble it. In this section, we give an overview of some of such scrambles.

**3.1. The Generalized Faure Sequence.** Tezuka [2] proposed the generalized Faure sequence, GFaure, with the  $j$ th dimension generator matrix  $\mathbf{C}^{(j)} = \mathbf{A}^{(j)} \mathbf{P}^{j-1}$  and the  $\mathbf{A}^{(j)}$  for  $j = 1, 2, \dots, s$  are arbitrary nonsingular lower triangular matrices over  $\mathbb{F}_p$ . A special case for  $\mathbf{A}^{(j)}$  is that all members are one for all dimensions [4].

**3.2. Random Linear (Digit) Scrambling.** After reviewing different versions of the Owen's method, Matoušek introduced a scramble matrix and a transfer vector for various dimensions [5]. The sequences obtained by Matoušek have the following general form:

$$\mathbf{x}_n = \left( \phi_p(\mathbf{A}^{(1)} \mathbf{P}^0 \mathbf{n} + \mathbf{g}_1), \phi_p(\mathbf{A}^{(2)} \mathbf{P}^1 \mathbf{n} + \mathbf{g}_2), \dots, \phi_p(\mathbf{A}^{(s)} \mathbf{P}^{s-1} \mathbf{n} + \mathbf{g}_s) \right). \quad (4)$$

For the random linear scrambling, the matrices  $\mathbf{A}^{(j)}$  and the vectors  $\mathbf{g}_j$  for  $j = 1, \dots, s$  are of the form

$$\mathbf{A}^{(j)} = \begin{pmatrix} h_{1,1} & 0 & 0 & 0 \\ h_{2,1} & h_{2,2} & 0 & 0 \\ h_{3,1} & h_{3,2} & h_{3,3} & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbf{g}_j = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{pmatrix}, \quad (5)$$

where the  $g_j$ 's and the  $h_{i,j}$  with  $i \geq j$  are chosen randomly and independently from  $\{0, 1, \dots, b_1\}$ , the  $h_{j,j}$ 's are chosen randomly and independently from  $\{1, 2, \dots, p-1\}$ .

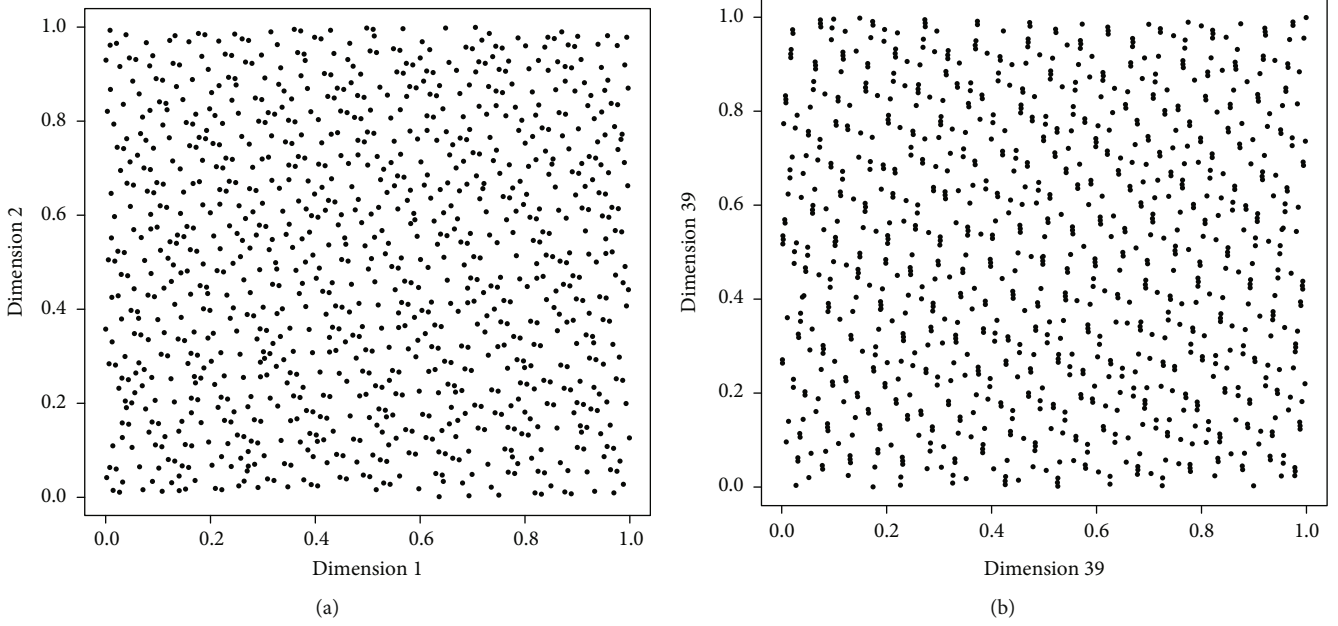


FIGURE 3: 1000 points from our scrambled Faure sequence.

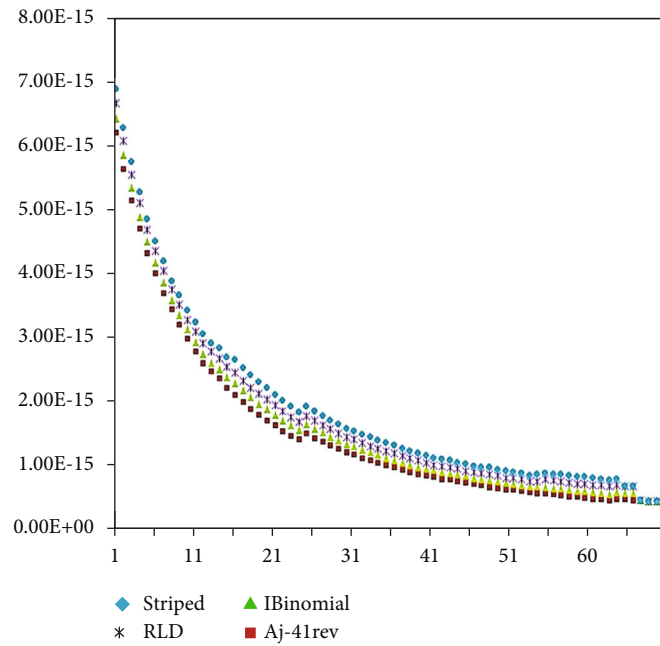


FIGURE 4:  $L_2$ -discrepancy for various randomized scramblings of a 40-dimensional Faure sequence.

Random linear digit method is the basis of other scrambles that will follow. Even the GFaure method is a subset of this method in which the members of the shift vectors are all zero.

3.3. *I-Binomial Scrambling.* A subset of the family of random linear scrambling methods is called left I-binomial scrambling [6]. Here, the  $A^{(j)}$  is defined as

$$A^{(j)} = \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 \\ h_2 & h_1 & 0 & 0 & 0 \\ h_3 & h_2 & h_1 & 0 & 0 \\ h_4 & h_3 & h_2 & h_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad (6)$$

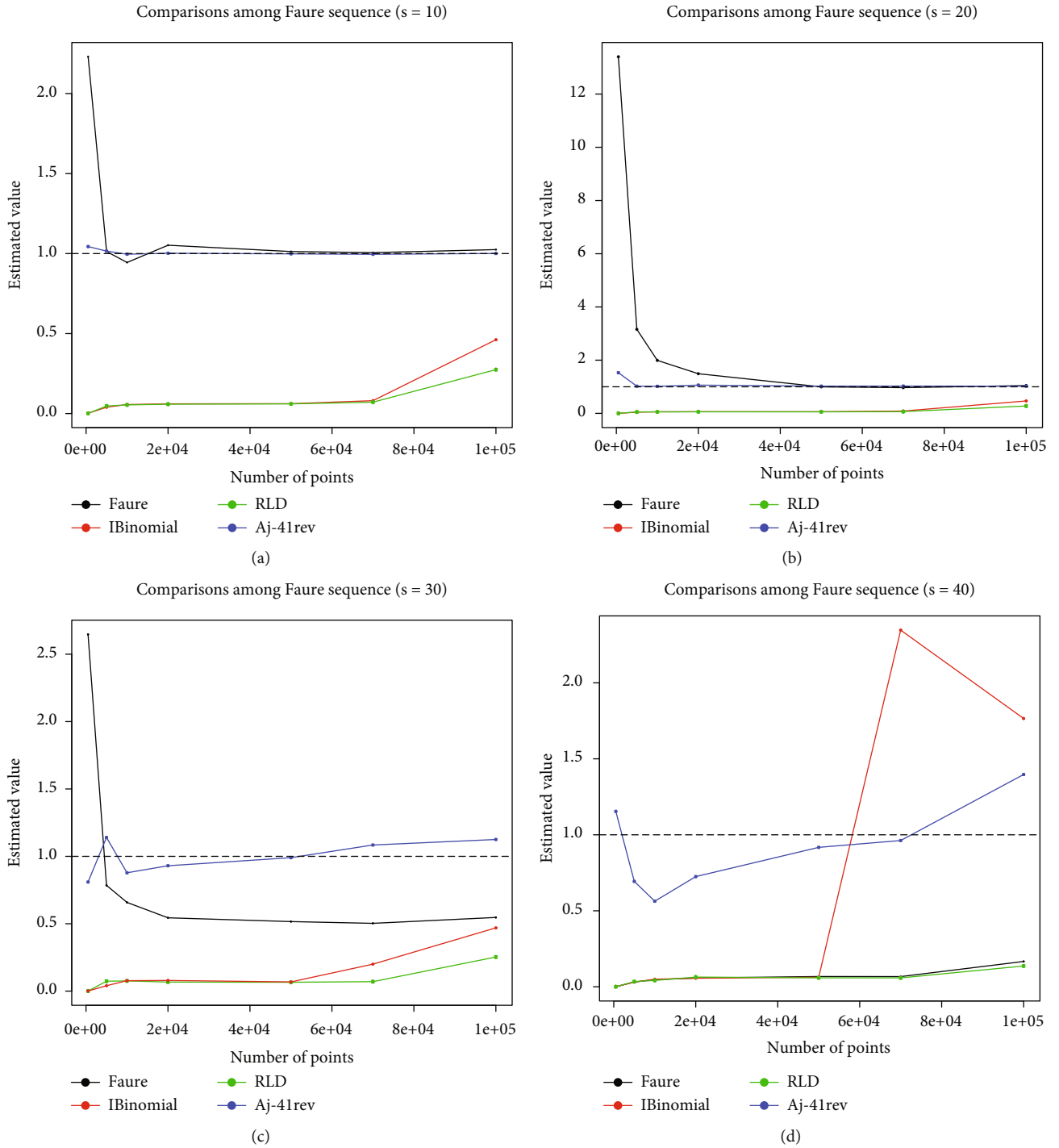


FIGURE 5: Estimates of the integral  $I_1(f)$  by using various Faure sequences.

where  $h_1$  is chosen randomly and independently from  $\{1, 2, \dots, p-1\}$  and also  $h_i$ 's ( $i > 1$ ) are chosen randomly and independently from  $\{0, 1, \dots, p-1\}$ .

3.4. *Striped Matrix Scrambling.* The scrambling matrix  $A^{(j)}$  for Striped Matrix Scrambling method, has the following form:

$$A^{(j)} = \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 \\ h_1 & h_2 & 0 & 0 & 0 \\ h_1 & h_2 & h_3 & 0 & 0 \\ h_1 & h_2 & h_3 & h_4 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad (7)$$

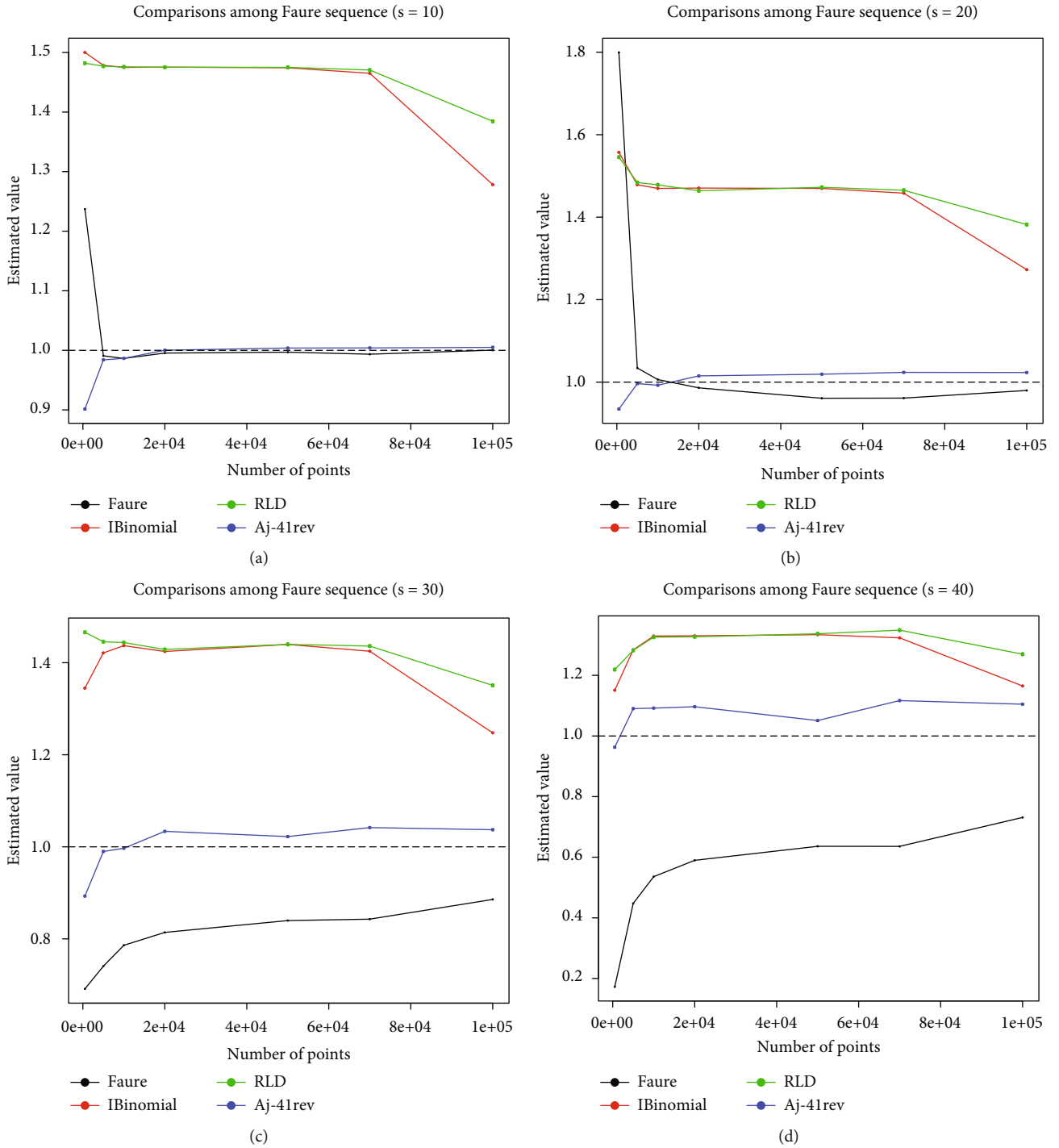


FIGURE 6: Estimates of the integral  $I_2(f)$  with  $a_i = 1$  by using various Faure sequences.

where the  $h_i$ 's are chosen randomly and independently from  $\{1, 2, \dots, p - 1\}$ . The different types of this scramble are examined in [7], only for problems in the first dimension. Of course, it does not say what changes should be made to the matrix for higher dimensions.

3.5. *Chi's Optimal Scramble.* When we use the I-binomial method to scramble the Faure sequence, the value of (number)  $h_1$  causes all the expansion digits of each number to be

replaced (permuted). Now, if we leave out the first digit, the value of (number)  $h_2$  causes all the remaining digits to be replaced (permuted).

So by cleverly selecting these two members, we can achieve better Faure sequences.

In [8], Chi has shown that the best choice for these two values can be obtained based on the primitive roots of the  $p$ . Finally, the Chi's optimal scramble matrix is as follows:

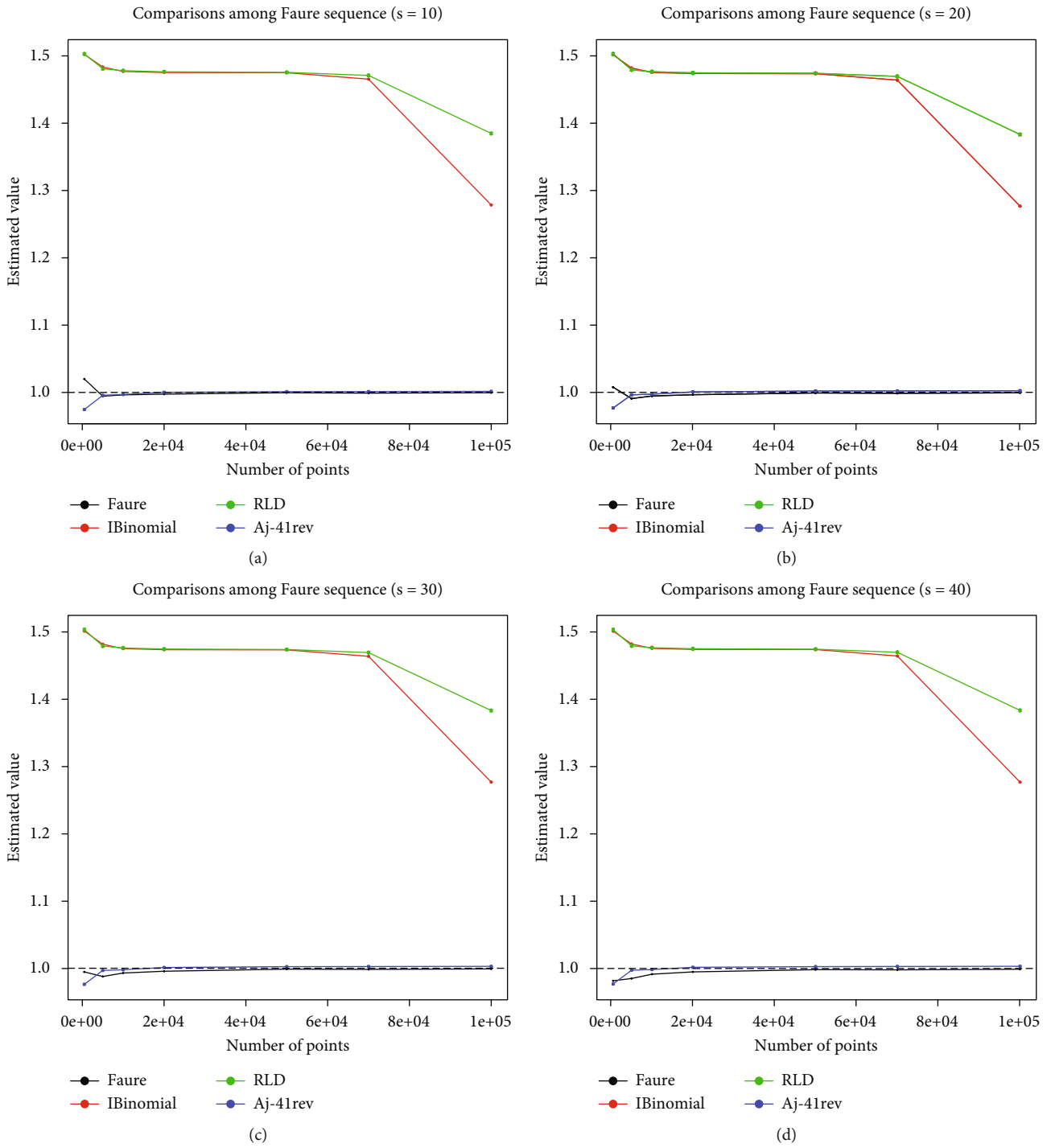


FIGURE 7: Estimates of the integral  $I_2(f)$  with  $a_i = i$  by using various Faure sequences.

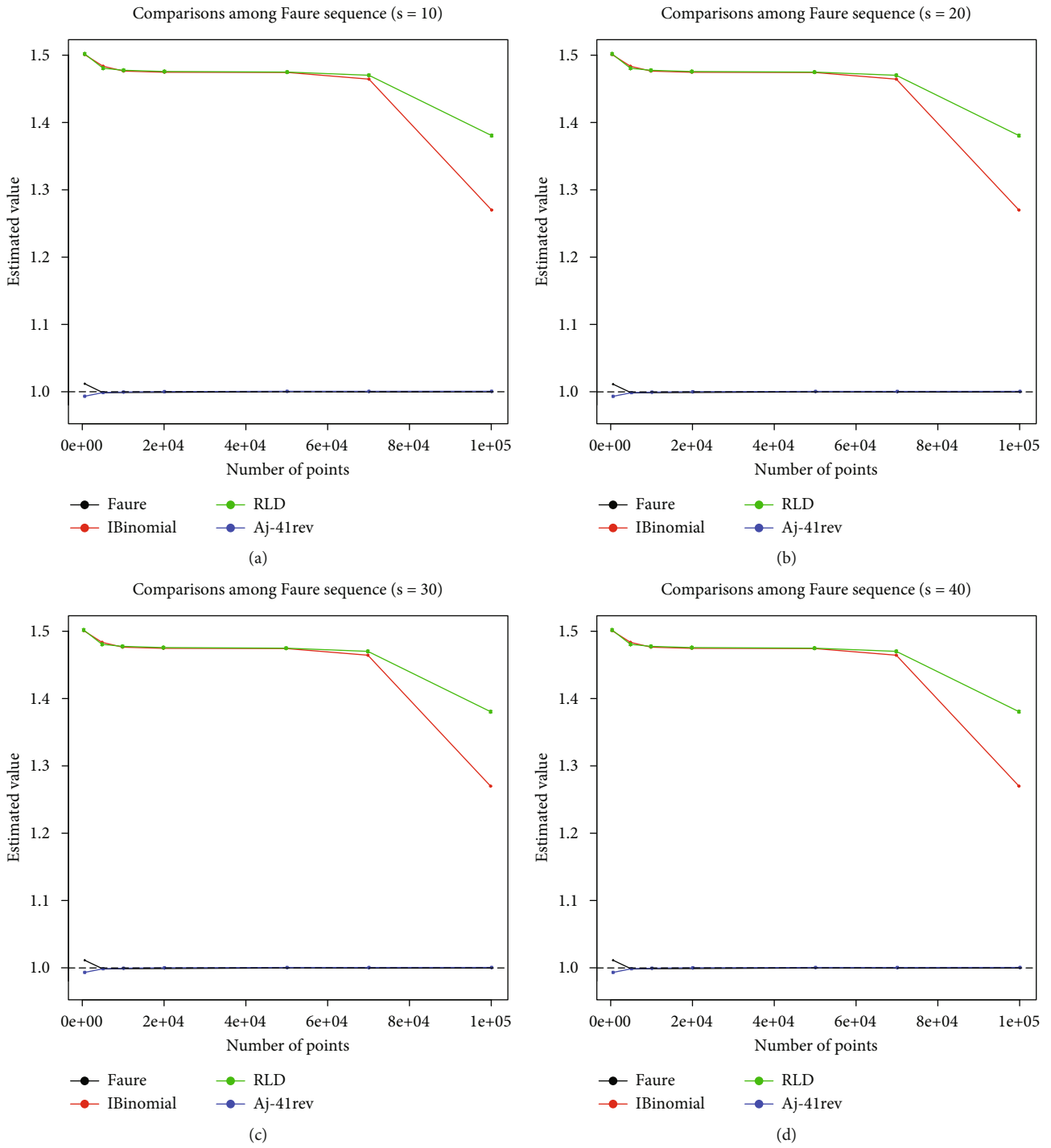


FIGURE 8: Estimates of the integral  $I_2(f)$  with  $a_i = i^2$  by using various Faure sequences.



$$\mathbf{A}^{(j)} = \begin{pmatrix} h_1^{j-1} & 0 & 0 & 0 \\ h_2 & h_1^{j-1} & 0 & 0 \\ 0 & h_2 & h_1^{j-1} & 0 \\ 0 & 0 & h_2 & h_1^{j-1} \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (8)$$

3.6. *Inverse Scrambling.* Fathi and Eskandari [9] adapted Chi's optimal matrix and introduced two kind of matrix  $\mathbf{A}^{(j)}$  as follows:

$$\mathbf{A}_1^{(j)} = \begin{pmatrix} d^{j-1} & 0 & 0 & 0 \\ 0 & d^{j-1} & 0 & 0 \\ 0 & 0 & d^{j-1} & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbf{A}_2^{(j)} = \begin{pmatrix} d^{j-1} & 0 & 0 & 0 \\ 0 & d^j & 0 & 0 \\ 0 & 0 & d^{j+1} & 0 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (9)$$

where  $d$  is chosen from  $\{1, 2, \dots, p-1\}$  for  $j = 1, \dots, s$ .

Based on nonlinear congruential method, they proposed another scrambling method for the Faure sequence for which the  $j$ th coordinate of the  $n$ th point has the general form

$$x_n^{(j)} = \phi_p \left( \Phi^{-1} \left( \mathbf{A}^{(j)} \Psi(\mathbf{P}^{j-1} \mathbf{n}) + \mathbf{g}_j \right) \right), \quad (10)$$

where  $\Phi(\mathbf{x})$  and  $\Psi(\mathbf{x})$  are bijections that map a digit vector  $\mathbf{x}$  to another digit vector.

#### 4. Scrambling Matrix with Irrational Members

In this section, corresponding to the method of random linear digits, we introduce a scrambling matrix that its members are a function of square root of base  $p$  (that is, they are irrational numbers).

After testing many functions, we found the following function that has the most performance:

$$\mathbf{A}^{(j)} = \begin{pmatrix} \sqrt{p}-1 & 0 & 0 & 0 \\ \sqrt{p}-1 & \sqrt{p}-1 & 0 & 0 \\ \sqrt{p}-1 & \sqrt{p}-1 & \sqrt{p}-1 & 0 \\ \sqrt{p}-1 & \sqrt{p}-1 & \sqrt{p}-1 & \sqrt{p}-1 \end{pmatrix}. \quad (11)$$

In (1), we introduced the common form of radical inverse function  $\phi$  for base  $p$ . Now, we define a new form of this function as follows:

$$\phi'_p(\mathbf{n}) = \frac{a_0}{p^m} + \frac{a_1}{p^{m-1}} + \dots + \frac{a_{m-1}}{p}. \quad (12)$$

TABLE 1: Estimates of  $I_1(f)$  by using Faure sequences.

Generator	$N$	$s=5$	$s=10$	$s=20$	$s=30$	$s=40$
Faure	500	1.2175	2.2301	13.402	2.6464	0.0001
IB	500	0.0017	0.0016	0.0014	0.0006	0.0002
RLD	500	0.0013	0.0013	0.0011	0.0006	0.0001
Aj-41rev	500	1.0392	1.0439	1.5286	0.8095	1.1542
Faure	5000	0.9803	1.0128	3.1585	0.7848	0.0314
IB	5000	0.0340	0.0397	0.0443	0.0402	0.0306
RLD	5000	0.0469	0.0475	0.0548	0.0737	0.0341
Aj-41rev	5000	1.0015	1.0154	1.0219	1.1401	0.6935
Faure	10000	0.9584	0.9445	1.9919	0.6583	0.0464
IB	10000	0.0569	0.0564	0.0593	0.0774	0.0496
RLD	10000	0.0539	0.0541	0.0566	0.0758	0.0418
Aj-41rev	10000	0.9992	0.9959	1.0182	0.8780	0.5630
Faure	20000	0.9943	1.0522	1.4909	0.5440	0.0575
IB	20000	0.0610	0.0607	0.0628	0.0791	0.0561
RLD	20000	0.0583	0.0585	0.0592	0.0670	0.0637
Aj-41rev	20000	0.9999	1.0021	1.0614	0.9301	0.7249
Faure	50000	0.9965	1.0127	0.9972	0.5159	0.0674
IB	50000	0.0617	0.0618	0.0621	0.0683	0.0628
RLD	50000	0.0604	0.0605	0.0582	0.0657	0.0587
Aj-41rev	50000	0.9984	0.9983	1.0241	0.9901	0.9173
Faure	70000	0.9964	1.0058	0.9680	0.5029	0.0669
IB	70000	0.0809	0.0806	0.0877	0.2000	2.3467
RLD	70000	0.0711	0.0713	0.0671	0.0702	0.0583
Aj-41rev	70000	0.9985	0.9954	1.0260	1.0841	0.9623
Faure	100000	1.0008	1.0248	1.0459	0.5466	0.1665
IB	100000	0.4610	0.4616	0.4667	0.4694	1.7654
RLD	100000	0.2742	0.2746	0.2772	0.2529	0.1366
Aj-41rev	100000	0.9979	1.0007	1.0215	1.1251	1.3969

We call this as *reverse radical inverse function*, and we denote the sequences that are made in this way, with the suffix "rev." Therefore, by combining the matrix  $\mathbf{A}^{(j)}$  and function  $\phi'$ , the general form of number  $n$  in dimension  $j$  will be as follows:

$$x_n^{(j)} = \phi'_p \left( \mathbf{A}^{(j)} \mathbf{P}^{j-1} \mathbf{n} \right). \quad (13)$$

So, for example, we denote the 40-dimensional Faure sequence generated on the base 41 by the scrambled matrix  $\mathbf{A}^{(j)}$  and the function  $\phi'$  with  $\mathbf{A}^{(j)}$ .41rev.

In the following sections, we have examined (studied) the quality of this sequence along with its performance compared to other sequences.

#### 5. Investigation of the Uniformity of Generated Sequences

5.1. *Two-Dimensional Projections.* The first step in evaluating the performance of a sequence is to see how the points in the 2D projections are distributed. From Figure 2, in these designs, the Faure sequence points are located within parallel

TABLE 2: Estimates of  $I_2(f)$  with  $a_i = 1$  by using Faure sequences.

Generator	$N$	$s = 5$	$s = 10$	$s = 20$	$s = 30$	$s = 40$
Faure	500	1.0759	1.2370	1.7994	0.6913	0.1727
IB	500	1.5015	1.5000	1.5572	1.3445	1.1510
RLD	500	1.5123	1.4820	1.5456	1.4661	1.2191
Aj-41rev	500	0.9879	0.9014	0.9347	0.8930	0.9629
Faure	5000	0.9963	0.9908	1.0342	0.7406	0.4475
IB	5000	1.4836	1.4785	1.4787	1.4212	1.2841
RLD	5000	1.4807	1.4770	1.4840	1.4455	1.2824
Aj-41rev	5000	0.9976	0.9840	0.9964	0.9898	1.0902
Faure	10000	0.9935	0.9864	1.0062	0.7860	0.5359
IB	10000	1.4769	1.4750	1.4697	1.4370	1.3296
RLD	10000	1.4779	1.4759	1.4784	1.4438	1.3261
Aj-41rev	10000	0.9990	0.9865	0.9925	0.9968	1.0920
Faure	20000	0.9945	0.9954	0.9861	0.8140	0.5897
IB	20000	1.4753	1.4755	1.4706	1.4244	1.3306
RLD	20000	1.4763	1.4752	1.4640	1.4288	1.3271
Aj-41rev	20000	1.0000	1.0002	1.0151	1.0336	1.0965
Faure	50000	0.9992	0.9969	0.9609	0.8397	0.6360
IB	50000	1.4750	1.4744	1.4697	1.4397	1.3342
RLD	50000	1.4755	1.4751	1.4725	1.4398	1.3372
Aj-41rev	50000	1.0015	1.0039	1.0192	1.0222	1.0509
Faure	70000	0.9983	0.9935	0.9613	0.8428	0.6458
IB	70000	1.4654	1.4650	1.4582	1.4249	1.3238
RLD	70000	1.4710	1.4705	1.4653	1.4362	1.3491
Aj-41rev	70000	1.0020	1.0040	1.0237	1.0418	1.1168
Faure	100000	0.9997	1.0003	0.9798	0.8856	0.7312
IB	100000	1.2785	1.2783	1.2727	1.2476	1.1649
RLD	100000	1.3848	1.3844	1.3821	1.3508	1.2698
Aj-41rev	100000	1.0027	1.0050	1.0233	1.0371	1.1048

TABLE 3: Estimates of  $I_2(f)$  with  $a_i = i$  by using Faure sequences.

Generator	$N$	$s = 5$	$s = 10$	$s = 20$	$s = 30$	$s = 40$
Faure	500	1.0243	1.0200	1.0077	0.9945	0.9817
IB	500	1.5010	1.5022	1.5031	1.5030	1.5023
RLD	500	1.5055	1.5029	1.5043	1.5051	1.5045
Aj-41rev	500	0.9951	0.9746	0.9768	0.9761	0.9770
Faure	5000	0.9972	0.9944	0.9908	0.9877	0.9851
IB	5000	1.4837	1.4835	1.4833	1.4833	1.4832
RLD	5000	1.4810	1.4809	1.4807	1.4806	1.4804
Aj-41rev	5000	0.9990	0.9957	0.9963	0.9968	0.9974
Faure	10000	0.9977	0.9964	0.9946	0.9929	0.9915
IB	10000	1.4770	1.4769	1.4768	1.4768	1.4766
RLD	10000	1.4781	1.4780	1.4779	1.4778	1.4777
Aj-41rev	10000	0.9998	0.9971	0.9973	0.9977	0.9984
Faure	20000	0.9979	0.9977	0.9965	0.9956	0.9949
IB	20000	1.4752	1.4753	1.4753	1.4752	1.4752
RLD	20000	1.4764	1.4763	1.4763	1.4762	1.4762
Aj-41rev	20000	1.0002	1.0002	1.0007	1.0012	1.0017
Faure	50000	0.9999	0.9996	0.9991	0.9987	0.9983
IB	50000	1.4750	1.4750	1.4750	1.4750	1.4749
RLD	50000	1.4755	1.4755	1.4755	1.4755	1.4755
Aj-41rev	50000	1.0009	1.0012	1.0019	1.0024	1.0027
Faure	70000	0.9993	0.9989	0.9986	0.9983	0.9980
IB	70000	1.4654	1.4654	1.4654	1.4654	1.4653
RLD	70000	1.4710	1.4710	1.4709	1.4709	1.4709
Aj-41rev	70000	1.0011	1.0013	1.0021	1.0026	1.0029
Faure	100000	0.9997	0.9996	0.9993	0.9990	0.9989
IB	100000	1.2785	1.2785	1.2785	1.2785	1.2784
RLD	100000	1.3848	1.3848	1.3848	1.3848	1.3848
Aj-41rev	100000	1.0013	1.0016	1.0024	1.0028	1.0032

lines, which shows that there is a linear correlation between points in successive dimensions. Also in these designs, we see a lot of empty spaces. Therefore, the distribution of good points has not been done. In Figure 3, we draw the same designs using the by  $A^{(j)}.41rev$ .

These figures show that two-dimensional projections of  $A^{(j)}.41rev$  sequence are better than two-dimensional projections of the original Faure sequence.

5.2. *Discrepancy.* One way to measure the quality of a sequence is to calculate its discrepancy [10]. Warnock shows that

$$(T_N)^2 = \frac{1}{N^2} \sum_{k=1}^N \sum_{m=1}^N \prod_{i=1}^s \left(1 - \max(x_k^{(i)}, x_m^{(i)})\right) - \frac{2^{1-s}}{N} \sum_{k=1}^N \prod_{i=1}^s \left(1 - x_k^{(i)^2}\right) + 3^{-s}. \tag{14}$$

where  $x_k^{(i)}$  is the  $i$ th component of the point  $x_k$ .

Figure 4 compares  $T_N$  between the original Faure sequence and the some scrambled Faure sequences that introduced in Section 3. From Figure 4, we see that the dis-

crepancy  $p$  diagram for our new sequence is at all points below the other sequences. This is a good indication of the high quality of our sequence points.

### 6. Numerical Integration

Another way to compare the quality of sequences is to use them to solve high-dimensional integration problems with numerical methods. Consider the following test integrals:

$$I_1(f) = \int_0^1 \dots \int_0^1 \prod_{i=1}^s \frac{\pi}{2} \sin(\pi x_i) dx_1 \dots dx_s = 1, \tag{15}$$

$$I_2(f) = \int_0^1 \dots \int_0^1 \prod_{i=1}^s \frac{|4x_i - 2| + a_i}{1 + a_i} dx_1 \dots dx_s = 1, \tag{16}$$

where the  $a_i$  are parameters. There are four choices of parameters as follows:

- (1)  $a_i = 0$  for  $1 \leq i \leq s$
- (2)  $a_i = 1$  for  $1 \leq i \leq s$

TABLE 4: Estimates of  $I_2(f)$  with  $a_i = i^2$  by using Faure sequences.

Generator	$N$	$s = 5$	$s = 10$	$s = 20$	$s = 30$	$s = 40$
Faure	500	1.0125	1.0115	1.0110	1.0108	1.0106
IB	500	1.5003	1.5005	1.5006	1.5006	1.5006
RLD	500	1.5017	1.5015	1.5015	1.5015	1.5015
Aj-41rev	500	0.9966	0.9936	0.9936	0.9936	0.9936
Faure	5000	0.9988	0.9986	0.9985	0.9984	0.9984
IB	5000	1.4838	1.4838	1.4838	1.4838	1.4838
RLD	5000	1.4810	1.4810	1.4810	1.4810	1.4810
Aj-41rev	5000	0.9993	0.9989	0.9989	0.9989	0.9989
Faure	10000	0.9991	0.9990	0.9989	0.9989	0.9988
IB	10000	1.4770	1.4770	1.4770	1.4770	1.4770
RLD	10000	1.4781	1.4781	1.4781	1.4781	1.4781
Aj-41rev	10000	1.0000	0.9996	0.9996	0.9996	0.9996
Faure	20000	0.9992	0.9991	0.9991	0.9991	0.9991
IB	20000	1.4753	1.4753	1.4753	1.4753	1.4753
RLD	20000	1.4764	1.4764	1.4764	1.4764	1.4764
Aj-41rev	20000	1.0003	1.0002	1.0003	1.0003	1.0003
Faure	50000	1.0000	1.0000	1.0000	1.0000	1.0000
IB	50000	1.4750	1.4750	1.4750	1.4750	1.4750
RLD	50000	1.4755	1.4755	1.4755	1.4755	1.4755
Aj-41rev	50000	1.0006	1.0006	1.0007	1.0007	1.0007
Faure	70000	0.9998	0.9997	0.9997	0.9997	0.9997
IB	70000	1.4655	1.4655	1.4655	1.4655	1.4655
RLD	70000	1.4710	1.4710	1.4710	1.4710	1.4710
Aj-41rev	70000	1.0006	1.0006	1.0007	1.0007	1.0007
Faure	100000	0.9998	0.9998	0.9998	0.9998	0.9998
IB	100000	1.2785	1.2785	1.2785	1.2785	1.2785
RLD	100000	1.3848	1.3848	1.3848	1.3848	1.3848
Aj-41rev	100000	1.0007	1.0007	1.0007	1.0008	1.0008

$$(3) a_i = i \text{ for } 1 \leq i \leq s$$

$$(4) a_i = i^2 \text{ for } 1 \leq i \leq s$$

Note that the most difficult case is when  $a = 0$ . Because in this case, the importance of all variables is the same and the superposition dimension is approximately the same as the truncation dimension. It is important to know that the larger  $a_i$ , the less important the variables are, and therefore, the effective dimension becomes smaller. The last three choices of the parameters will be considered here [11].

In numerical solution of problems with qMC methods, an accepted procedure is to omit the starting points of the sequence. For example, Fox [12] has suggested that we consider the starting point of the sequence as  $n = QS^4 - 1$ . Although, this may lead to better results, note that with this selection, a large amount of points must be omitted. For example, for  $s = 40$ , we have to skip the initial 2825760 points with this formula, which is practically impossible. We found in our research that if we select the  $p$  as the starting point, it will significantly improve the results. We will probably get the best result when we start from the  $(p^2 + 1)$ th point.

Therefore, we skip the first 41 points and start  $n = 42$  in our calculations. For comparison purposes, we present numerical results for original Faure (Faure), our sequence (Aj-41rev), and two types of scrambled Faure sequences, I-binomial (IB) and random linear digits (RLD).

Now, we compare the numerical results of different scrambled Faure sequences presented in this paper. The estimated values for the test functions are given in Figures 5–8. These figures show that our proposed scramble has a very acceptable convergence compared to other scrambles.

An observation is that estimated values by the matrix Aj-41rev very close to the actual value. This can be seen in Tables 1–4. The estimation error obtained with this scramble in dimension 40, for the function (16) with parameters  $a_i = 1$ ,  $a_i = i$ , and  $a_i = i^2$  are at most 11.68%, 0.32%, and 0.64%, respectively. However, there are some exceptions. For example, in the first function, when the number of dimensions increases, the accuracy of the estimation decreases. So that, for the dimension 40, the maximum estimated relative error value is 43.7%.

## 7. Conclusion

We studied the original Faure sequence and some of its recent years introduced scrambles. Then, we introduced a new scrambling matrix based on irrational numbers that its elements are function of square root of base  $p$ . In Figures 5–8, we have shown that this modified scrambled Faure sequence provides better results than the previous versions of its scrambles. Also, we presented that this modified scrambled Faure sequence has greatly improved the distribution of points. The 2D designs confirm this claim, good two-dimensional projections in successive dimensions. As mentioned in the previous section, using this scramble leads to very small estimated relative errors that can often be ignored. In our next research, we will use deterministic scrambling matrices based on another irrational numbers and primitive roots. The results proved the improvements of accuracy using our new scramble.

## Data Availability

The data is contained in the article itself.

## Conflicts of Interest

The authors declares that there is no conflict of interest regarding the publication of this paper.

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