# Bayes Estimators of Exponentiated Inverse Rayleigh Distribution using Lindleys Approximation

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*Authors' contributions*

*This work was carried out in collaboration between all authors. Author TMA designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author BOS and Author KTU managed the analyses of the study and managed the literature searches. All authors read and approved the final manuscript.*

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## Abstract

In this paper, Bayes estimators of the unknown shape and scale parameters of the Exponentiated Inverse Rayleigh Distribution (EIRD) have been derived using both the frequentist and bayesian methods. The Bayes theorem was adopted to obtain the posterior distribution of the shape and scale parameters of an Exponentiated Inverse Rayleigh Distribution (EIRD) using both conjugate and non-conjugate prior distribution under different loss functions (such as Entropy Loss Function, Linex Loss Function and Scale Invariant Squared Error Loss Function). The posterior distribution derived for both shape and scale parameters are intractable and a Lindley approximation was adopted to obtain the parameters of interest. The loss function were employed to obtain the estimates for both scale and shape parameters with an assumption that the both scale and

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shape parameters are unknown and independent. Also the Bayes estimate for the simulated datasets and real life datasets were obtained. The Bayes estimates obtained under different loss functions are close to the true parameter value of the shape and scale parameters. The estimators are then compared in terms of their Mean Square Error (MSE) using R programming language. We deduce that the MSE reduces as the sample size (n) increases.

# 1   Introduction

Rayleigh distribution originated from a two parameters Weibull distribution and it's a suitable model for modelling life-time data sets. Let the random variable T follows a Rayleigh distribution, then the random variable $x = \frac{1}{T}$ has an inverse Rayleigh distribution (IRD). The inverse Rayleigh distribution (IRD) was introduced by [1] for modeling realibility and survival data sets. [2] studied some properties of IRD and [3] discussed the properties and maximum likelihood estimation of the scale parameter of IRD. The variance and the higher order moments of this distribution do not exist. The reliability sampling plans of IRD was carried out by [4]. The probability density function (PDF) of the one parameter IRD

$$f(x, \sigma) = \frac{2\sigma}{x^3} e^{\left(-\frac{\sigma}{x}\right)}, \qquad \sigma > 0, x > 0. \tag{1.1}$$

The closed-form expressions for the mean, harmonic mean, geometric mean, mode and the median of IRD was discussed by [5]. The estimation of the parameter $\sigma$ using both different classical and Bayesian estimation methods was carried out by [5] and [6]. In recent years, attention has been shifted to the generalization of probability distribution theory, most applied in reliability estimation [7, 8, 9, 10]. The transmuted Rayleigh distribution and transmuted generalized Rayleigh distribution were developed by [11, 12] respectively. [13] and [14] proposed a Beta Inverse Rayleigh. The exponentiated inverse Rayleigh distribution (EIRD) also known as a life time distribution was introduced by [15]. This distribution can be adopted for reliability estimation and statistical quality control. The probability density function (pdf) of EIRD is written as

$$f(x) = \frac{2\alpha\sigma^2}{x^3} e^{-\left(\frac{\sigma}{x}\right)^2} \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha-1}; \qquad \alpha > 0, \sigma > 0, x > 0 \tag{1.2}$$

where $\alpha$ is the shape parameter and $\sigma$ is the scale parameter and the exponential inverse exponential distribution is denoted as $\text{EIED}(\alpha, \sigma)$. The inverse Rayleigh distribution is the particular case of (1.2) for $\alpha = 1$. The cumulative density function (cdf) is defined as

$$F(x) = 1 - \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha} \qquad \alpha > 0, \sigma > 0, x > 0 \tag{1.3}$$

Fig (1), (2), (3) and (4) shows the PDF, CDF, reliability function and hazard function of EIRD for various values of both shape and scale parameters.

The reliability function is given by

$$R(x) = \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha} \qquad \alpha > 0, \sigma > 0, x > 0 \tag{1.4}$$

and the hazard function is

$$h(x) = \frac{2\alpha\sigma^2}{x^3} e^{-\left(\frac{\sigma}{x}\right)^2} \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{-1}; \qquad \alpha > 0, \sigma > 0, x > 0 \tag{1.5}$$

In this article, we propose the Bayes estimators for shape and scale parameters of an EIRD under
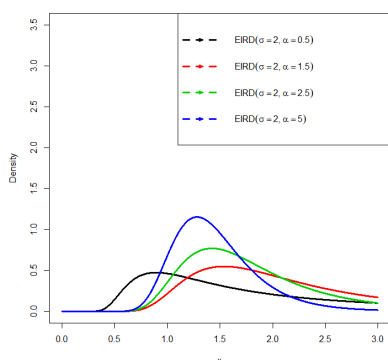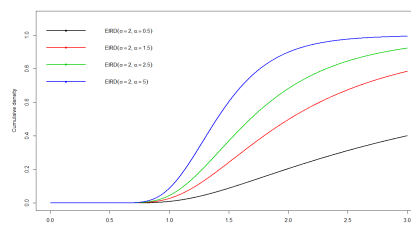
Fig. 1. PDF of an EIRD
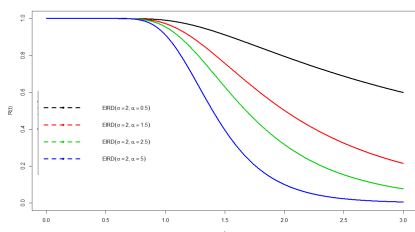


Fig. 2. CDF of an EIRD


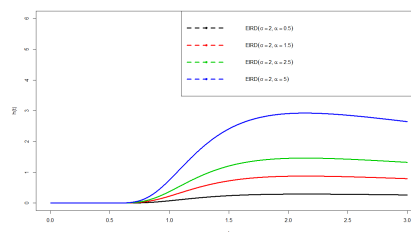
Fig. 3. Reliability graph of an EIRD



Fig. 4. Hazard graph of an EIRD

the Entropy Loss Function, Linex Loss Function and Scale Invariant Squared Error Loss Function given that the scale and shape parameters are unknown. In Sections 2, we discuss the estimation of the shape and scale parameters. In Section 3, numerical results are presented for both the simulated and real-life data on survival times of patients with breast cancer, and Section 4 contains the conclusion.

# 2 Materials and Methods

## 2.1 Maximum likelihood function

Let x $= (x_1, x_2, \ldots, x_n)$ be a random variable drawn from EIRD with size n. The likelihood function for the given random sample can be expressed as

$$L(x/\sigma, \alpha) = 2^n \sigma^n \alpha^n \prod_{i=1}^{n} x^{-3} e^{-\sum \left(\left(\frac{\sigma}{x}\right)^2\right)} \prod_{i=1}^{n} \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha-1} \tag{2.1}$$

The log-likelihood function of (2.1) is

$$\ln L(x, \sigma, \alpha) = n \log \alpha + 2n \log \sigma - 3 \sum_{i=1}^{n} (\log x) -$$
$$\sum_{i=1}^{n} \left(\frac{\sigma}{x}\right)^2 + (\alpha + 1) \sum_{i=1}^{n} \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right) \tag{2.2}$$

The maximum likelihood estimator of the shape and scale parameters $\sigma$ and $\alpha$ is obtained by differentiating the (2.2) on parameters $\sigma$ and $\alpha$. The maximum likelihood differential equations are:

$$\frac{\partial \ln L(x/\sigma)}{\partial \sigma} = \frac{2n}{\sigma} - 2\sigma \sum_{i=1}^{n} \frac{1}{x^2} + 2\sigma(\alpha - 1) \sum_{i=1}^{n} \frac{e^{-(\frac{\sigma}{x})^2}}{x^2 \left(1 - e^{-(\frac{\sigma}{x})^2}\right)} \tag{2.3}$$

$$\frac{\partial \ln L(x/\alpha)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left(1 - e^{-(\frac{\sigma}{x})^2}\right) \tag{2.4}$$

equating $(2.4) = 0$, we will obtain the maximum likelihood estimate of $\hat{\alpha}$

$$\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n} \ln\left(1 - e^{-(\frac{\sigma}{x})^2}\right)} \tag{2.5}$$

substituting (2.5) in (2.3), we obtained an expression for the parameter $\sigma$ and equating to zero.

$$\frac{2n}{\sigma} - 2\sigma \sum_{i=1}^{n} \frac{1}{x^2} + 2\sigma \left(\frac{-n}{\sum_{i=1}^{n} \ln\left(1 - e^{-(\frac{\sigma}{x})^2}\right)} - 1\right) \sum_{i=1}^{n} \frac{e^{-(\frac{\sigma}{x})^2}}{x^2 \left(1 - e^{-(\frac{\sigma}{x})^2}\right)} = 0 \tag{2.6}$$

from (2.6), it's clear that the equation is not in explicit form, so to obtain the estimates of parameters $\sigma$ and $\alpha$ we solve the nonlinear equations (2.3) and (2.4) simultaneously. The estimated value for parameters $\sigma$ and $\alpha$ can be obtained numerically using an iterative approach known as Newton Raphson method [16, 17, 18]. The elements of the Fisher information matrix for the parameter $\sigma$ and $\alpha$ can be expressed as

$$J_k = \begin{bmatrix} \frac{\partial^2 l(\sigma,\alpha)}{\partial \sigma^2} & \frac{\partial^2 l(\sigma,\alpha)}{\partial \sigma \partial \alpha} \\ \frac{\partial^2 l(\sigma,\alpha)}{\partial \sigma \partial \alpha} & \frac{\partial^2 l(\sigma,\alpha)}{\partial \alpha^2} \end{bmatrix} \tag{2.7}$$

The Jacobian matrix must be a non-singular symmetric matrix so its inverse must exist. So, using the Newton Raphson method we have

$$\begin{bmatrix} \sigma_{k+1} \\ \alpha_{k+1} \end{bmatrix} = \begin{bmatrix} \sigma_k \\ \alpha_k \end{bmatrix} - J_k^{-1} \begin{bmatrix} \frac{\partial l(\sigma,\alpha)}{\partial \sigma} \\ \frac{\partial l(\sigma,\alpha)}{\partial \alpha} \end{bmatrix} \tag{2.8}$$

with error term $\epsilon$ being the absolute differences between the new and the previous value of $\sigma$ and $\alpha$ in the iterative algorithm. That is

$$\epsilon \begin{bmatrix} \epsilon_{k+1}(\sigma) \\ \epsilon_{k+1}(\alpha) \end{bmatrix} = \left[ \begin{bmatrix} \sigma_{k+1} \\ \alpha_{k+1} \end{bmatrix} - \begin{bmatrix} \sigma_k \\ \alpha_k \end{bmatrix} \right] \tag{2.9}$$

where $\sigma_k$ and $\alpha_k$ are the initial values of $\sigma$ and $\alpha$ respectively.

## 2.2 The Bayesian estimation of the parameters of EIRD

In the estimation of EIRD parameters under Bayesian method, three types of loss function were considered. The first is LINEX loss function (LLF) which is also known as linear-exponential loss function which is asymmetric. [19] introduced the LLF and these loss function have been adopted by several authors such as [20, 21, 22, 23, 24, 25] among many. The LLF rises approximately exponentially on one side of zero and approximately linearly on the other side [26]. The second is Entropy loss function (ELF) which was introduced by [27]. The ELF is an asymmetric loss function which have been used by several authors like [28, 29, 30, 31]. These authors used ELF in its original form by specifying c to be equal to 1. The third loss function is scale invariant squared error loss

function (SISLF) was introduced by [32] and it is also known as De-Groot loss function. The SISLF has been used by several author such as [33, 34] The LLF can be expressed as

$$L_{LLF}\kappa(e^{\tau(\hat{\theta}-\theta)} - \tau(\hat{\theta}-\theta) - 1) \qquad \kappa > 0, \tau \neq 0 \tag{2.10}$$

where $\tau_1$ and $\kappa$ are the scale and shape parameters of the LLF. In this study, we assume that $\kappa=$ 1. Bayes estimator of the LLF is the value $\hat{\theta}$ that minimizes (2.10) [[35]].

$$\hat{\theta} = -\frac{1}{\tau}\ln\left(E_\theta\left[e^{-\tau\theta}\right]\right) \tag{2.11}$$

provided that $E_\theta\left[e^{-\tau\theta}\right]$ exits.
The ELF is defined as

$$L_{ELF}\left(\frac{\hat{\theta}}{\theta}\right)^p \propto \left[\left(\frac{\hat{\theta}}{\theta}\right)^p - p\log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right] \tag{2.12}$$

where $p > 0$. The minimum occurred at $\hat{\theta} = \theta$. If p = 1, as used by [28, 29], then (2.12) can be expressed as

$$L_{ELF}\left(\frac{\hat{\theta}}{\theta}\right) = \left[\left(\frac{\hat{\theta}}{\theta}\right) - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right] \tag{2.13}$$

The Bayes estimator of the ELF is the value $\hat{\theta}$ that minimizes (2.13) and can be expressed as

$$\hat{\theta} = \left[E\left(\theta^{-1}\right)\right]^{-1} \tag{2.14}$$

[32] defined SISLF as

$$L_{SISLF}(\hat{\theta}, \theta) = \left(\frac{\theta - \hat{\theta}}{\theta}\right)^2 \tag{2.15}$$

The Bayes' estimates can be expressed as

$$\hat{\theta}_{SISLF} = \frac{E\left(\frac{1}{\theta}\right)}{E\left(\frac{1}{\theta^2}\right)} \tag{2.16}$$

To obtain the Bayes' estimate of $\alpha$ and $\sigma$, we need prior distribution for the parameters $\alpha$ and $\sigma$. For the two parameters we consider a non-informative prior for the shape parameter and a natural conjugate prior for the scale parameter (with the assumption that the shape parameter is known). Thus the proposed prior distribution for the parameters $\alpha$ and $\sigma$ can be expressed as

$$\pi_1(\alpha) = \frac{1}{\alpha} \qquad \alpha > 0 \tag{2.17}$$

$$\pi_2(\sigma) = \frac{b^a}{\Gamma(a)}\sigma^{a-1}e^{-b\sigma} \qquad a > 0, b > 0, \sigma > 0 \tag{2.18}$$

The joint prior distribution for the parameters $\alpha$ and $\sigma$ is

$$\pi(\alpha, \sigma) = \frac{b^a}{\alpha\Gamma(a)}\sigma^{a-1}e^{-b\sigma} \qquad a > 0, b > 0, \alpha > 0, \sigma > 0 \tag{2.19}$$

To obtain the posterior distribution of the parameters $\alpha$ and $\sigma$ we substitute (2.1) and (2.19) in (2.20)

$$\Pr(\alpha, \sigma|x) \propto \frac{L(\alpha, \sigma|x)\pi(\alpha, \sigma)}{\int_0^\infty \int_0^\infty L(\alpha, \sigma|x)\pi(\alpha, \sigma)\partial\alpha\partial\sigma} \tag{2.20}$$

$$\Pr(\alpha, \sigma|x) = \frac{\frac{b^a 2^n \sigma^n \alpha^n}{\alpha\Gamma(a)}\sum_{i=1}^n x^{-3}e^{-\sum(\frac{\sigma}{x})^2}\prod_{i=1}^n\left(1 - e^{-(\frac{\sigma}{x})^2}\right)^{\alpha-1}\sigma^{a-1}e^{-b\sigma}}{\int_0^\infty \int_0^\infty \frac{b^a 2^n \sigma^n \alpha^n}{\alpha\Gamma(a)}\sum_{i=1}^n x^{-3}e^{-\sum(\frac{\sigma}{x})^2}\prod_{i=1}^n\left(1 - e^{-(\frac{\sigma}{x})^2}\right)^{\alpha-1}\sigma^{a-1}e^{-b\sigma}\partial\alpha\partial\sigma} \tag{2.21}$$

It's noted that the posterior distribution (2.21) takes a ratio form that involves an integration in the denominator and that the denominator can't be reduced to a closed form. Thus to estimate the posterior distribution (2.21) will be difficult. In other to estimate the posterior distribution we will adopt the Lindley's approximation suggested by [36] which treats the ratio of the integrals as a whole which results to a single numerical result. Many authors have employed this approximation for obtaining the Bayes estimators for some lifetime distributions; see among others, [31, 30, 37, 38] In this work, we compute $E(\theta_i|x)$ and $E(\theta_i^2|x)$ in order to find the variance estimates given by

$$Var(\theta_i|x) = E(\theta_1^2|x) - (E(\theta_i|x))^2 \qquad i = 1, 2 \tag{2.22}$$

where $\theta_1 = \sigma$ and $\theta_2 = \alpha$ If n is sufficiently large, according to [36], any ratio of the integral of the form

$$I(x) = E[u(\sigma, \alpha)] = \frac{\int \int u(\sigma, \alpha) e^{l(\sigma, \alpha) + \rho(\sigma, \alpha)} d\sigma d\alpha}{\int \int e^{l(\sigma, \alpha) + \rho(\sigma, \alpha)} d\sigma d\alpha} \tag{2.23}$$

where $u(\sigma, \alpha)$ is a function of $\sigma$ and $\alpha$ only, $l(\sigma, \alpha)$ is the log-likelihood and $\rho(\sigma, \alpha)$ is the log of prior distribution $\pi(\sigma, \alpha)$ Thus, for the unknown parameter $\sigma$ the Lindley's approximation is

$$E[u(\sigma, \alpha)|\underline{x} = u(\hat{\sigma}, \hat{\alpha}) + \frac{1}{2}(u_{11}\phi_{11}) + \rho_1 u_1 \phi_{11} + \frac{1}{2}(L_{30}u_1\phi_{11}^2) + \frac{1}{2}(L_{12}u_1\phi_{11}\phi_{22}) \tag{2.24}$$

where $u(\hat{\sigma}, \hat{\alpha}) = \frac{1}{\hat{\sigma}}$ Also, for the unknown parameter $\alpha$ the Lindley's approximation is

$$E[u(\sigma, \alpha)|\underline{x} = u(\hat{\sigma}, \hat{\alpha}) + \frac{1}{2}(u_{22}\phi_{22}) + \rho_2 u_2 \phi_{22} + \frac{1}{2}(L_{03}u_2\phi_{22}^2) + \frac{1}{2}(L_{21}u_2\phi_{11}\phi_{22}) \tag{2.25}$$

where $u(\hat{\sigma}, \hat{\alpha}) = \frac{1}{\hat{\alpha}}$
All the quantities in the above expression of I(x) have the following representations:

$$L_{ij} = \frac{\partial^{i+j} l(\sigma, \alpha)}{\partial \sigma^i \partial \alpha^j} \qquad i, j = 0, 1, 2, 3$$

$$L_{12} = \frac{\partial^3 l(\sigma, \alpha)}{\partial \sigma \partial \alpha^2} = 0$$

$$L_{21} = \frac{\partial^3 l(\sigma, \alpha)}{\partial \sigma^2 \partial \alpha} = 2 \sum_{i=1}^n \frac{e^{\left(\frac{\sigma}{x}\right)^2}}{x^2 \left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)} - 4\sigma^2 \sum_{i=1}^n \frac{e^{\left(\frac{\sigma}{x}\right)^2}}{x^4 \left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)}$$

$$-4\sigma^2 \sum_{i=1}^n \frac{\left(e^{\left(\frac{\sigma}{x}\right)^2}\right)^2}{x^4 \left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)}$$

$$L_{30} = \frac{\partial^3 l(\sigma, \alpha)}{\partial \sigma^3} = \frac{2n}{\alpha^3}$$

$$L_{20} = \frac{\partial^2 l(\sigma, \alpha)}{\partial \sigma^2} = -\frac{2n}{\sigma^2} - 2 \sum_{i=1}^n \left(\frac{1}{x}\right)^2 + 2(\alpha - 1) \sum_{i=1}^n \frac{e^{\left(\frac{\sigma}{x}\right)^2}}{x^2 \left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)}$$

$$-4\sigma^2 \sum_{i=1}^n \frac{e^{\left(\frac{\sigma}{x}\right)^2}}{\left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)} - 4(\alpha - 1)\sigma^2 \sum_{i=1}^n \frac{e^{\left(\frac{\sigma}{x}\right)^2}}{x^2 \left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)}$$

$$-4\sigma^2 \sum_{i=1}^n \frac{e^{\left(\frac{\sigma}{x}\right)^2}}{x^4 \left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)} - 4(\alpha - 1)\sigma^2 \sum_{i=1}^n \frac{\left(e^{\left(\frac{\sigma}{x}\right)^2}\right)^2}{x^2 \left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)} - 4\sigma^2 \sum_{i=1}^n \frac{e^{\left(\frac{\sigma}{x}\right)^2}}{x^4 \left(1 - e^{\left(\frac{\sigma}{x}\right)^2}\right)}$$

$$L_{02} = \frac{\partial^2 l(\sigma, \alpha)}{\partial \alpha^2} = -\frac{n}{\alpha^2}$$

$$L_{30} = \frac{\partial^3 l(\sigma, \alpha)}{\partial \alpha^3} = \frac{2n}{\alpha^3}$$

$$\phi_{11} = -\frac{1}{L_{20}}$$

and

$$\phi_{22} = -\frac{1}{L02}$$

The values of the Bayes estimates of parameters $\sigma$ and $\alpha$ can now be obtained.

a.  Case of the LINEX loss function (LLF)

i  For the parameter $\sigma$, let $u(\hat{\sigma}, \hat{\alpha}) = e^{-k\hat{\sigma}}$ then $u_1 = -ke^{-k\hat{\sigma}}$ and $u_{11} = k^2 e^{-k\hat{\sigma}}$, $u_2 = u_{22} = 0$

$$\hat{\sigma}_{LLF} = -\frac{1}{k}\left[e^{-k\hat{\sigma}} + \frac{1}{2}u_{11}\phi_{11} + \rho_1 u_1 \phi_{11} + \frac{1}{2}L_{30}u_1\phi_{11}^2 + \frac{1}{2}L_{12}u_1\phi_{11}\phi_{22}\right] \quad (2.26)$$

ii  For the parameter $\alpha$, let $u(\hat{\sigma}, \hat{\alpha}) = e^{-k\hat{\alpha}}$ then $u_2 = -ke^{-k\hat{\alpha}}$ and $u_{22} = k^2 e^{-k\hat{\alpha}}$, $u_1 = u_{11} = 0$

$$\hat{\alpha}_{LLF} = -\frac{1}{k}\left[e^{-k\hat{\alpha}} + \frac{1}{2}u_{22}\phi_{22} + \rho_2 u_2 \phi_{22} + \frac{1}{2}L_{03}u_2\phi_{22}^2 + \frac{1}{2}L_{21}u_2\phi_{11}\phi_{22}\right] \quad (2.27)$$

b.  Case of the Entropy loss function (ELF)

i  For the parameter $\sigma$, let $u(\hat{\sigma}, \hat{\alpha}) = \frac{1}{\sigma}$ then $u_1 = -\frac{1}{\sigma^2}$ and $u_{11} = \frac{2}{\sigma^3}$, $u_2 = u_{22} = 0$

$$\hat{\sigma}_{ELF} = \left[\frac{1}{\hat{\sigma}} + \frac{1}{2}u_{11}\phi_{11} + \rho_1 u_1 \phi_{11} + \frac{1}{2}L_{30}u_1\phi_{11}^2 + \frac{1}{2}L_{12}u_1\phi_{11}\phi_{22}\right]^{-1} \quad (2.28)$$

ii  For the parameter $\alpha$, let $u(\hat{\sigma}, \hat{\alpha}) = \frac{1}{\hat{\alpha}}$ then $u_2 = -\frac{1}{\alpha^2}$ and $u_{22} = \frac{2}{\alpha^3}$, $u_1 = u_{11} = 0$

$$\hat{\alpha}_{ELF} = \left[\frac{1}{\hat{\alpha}} + \frac{1}{2}u_{22}\phi_{22} + \rho_2 u_2 \phi_{22} + \frac{1}{2}L_{03}u_2\phi_{22}^2 + \frac{1}{2}L_{21}u_2\phi_{11}\phi_{22}\right]^{-1} \quad (2.29)$$

c.  Case of the scale invariant squared error loss function (SISLF)

i  For the parameter $\sigma$, let $u(\hat{\sigma}, \hat{\alpha}) = \frac{1}{\sigma}$ then $u_1 = -\frac{1}{\sigma^2}$ and $u_{11} = \frac{2}{\sigma^3}$, $u_2 = u_{22} = 0$ and also let $u^*(\hat{\sigma}, \hat{\alpha}) = \frac{1}{\sigma^2}$ then $u_1^* = -\frac{2}{\sigma^3}$ and $u_{11}^* = \frac{6}{\sigma^4}$, $u_2^* = u_{22}^* = 0$

$$\hat{\sigma}_{SISLF} = \frac{\left[\frac{1}{\hat{\sigma}^2} + \frac{1}{2}u_{11}^*\phi_{11} + \rho_1 u_1^*\phi_{11} + \frac{1}{2}L_{30}u_1^*\phi_{11}^2 + \frac{1}{2}L_{12}u_1^*\phi_{11}\phi_{22}\right]}{\left[\frac{1}{\hat{\sigma}} + \frac{1}{2}u_{11}\phi_{11} + \rho_1 u_1\phi_{11} + \frac{1}{2}L_{30}u_1\phi_{11}^2 + \frac{1}{2}L_{12}u_1\phi_{11}\phi_{22}\right]} \quad (2.30)$$

ii  For the parameter $\alpha$, Let $u^*(\hat{\sigma}, \hat{\alpha}) = \frac{1}{\hat{\alpha}^2}$ then $u_2^* = -\frac{2}{\alpha^3}$ and $u_{22}^* = \frac{6}{\alpha^4}$, $u_1^* = u_{11}^* = 0$ and also let $u(\hat{\sigma}, \hat{\alpha}) = \frac{1}{\hat{\alpha}}$ then $u_2 = -\frac{1}{\alpha^2}$ and $u_{22} = \frac{2}{\alpha^3}$, $u_1 = u_{11} = 0$

$$\hat{\alpha}_{SISLF} = \frac{\left[\frac{1}{\hat{\alpha}^2} + \frac{1}{2}u_{22}^*\phi_{22} + \rho_2 u_2^*\phi_{22} + \frac{1}{2}L_{03}u_2^*\phi_{22}^2 + \frac{1}{2}L_{21}u_2^*\phi_{11}\phi_{22}\right]}{\left[\frac{1}{\hat{\alpha}} + \frac{1}{2}u_{22}\phi_{22} + \rho_2 u_2\phi_{22} + \frac{1}{2}L_{03}u_1\phi_{22}^2 + \frac{1}{2}L_{21}u_2\phi_{11}\phi_{22}\right]} \quad (2.31)$$

# 3 Analysis

## 3.1 Monte Carlo Simulation

In this section, we simulated a random sample of sizes $n = 30, 50, 100$ and $200$ from an EIRD with parameters $\sigma = 0.5, 1.5$ and $2$, $\alpha = 0.5$ and $0.8$, $k = 1$ and $2$. The results were replicated 10,000 times and the average result were presented in the tables using R Software. Monte Carlo method is any computational approach psuedo-random number solve a mathematical problems as defined by [39]. Thus the numerical approach follows

1. For known parameters values $(\sigma, \alpha)$, we simulated a random sample of size of n from EIRD using the quantile function of the EIRD distribution is given by

$$Q(U) = \frac{\sigma}{\sqrt{-\log\left(1 - (1-U)^{\frac{1}{\alpha}}\right)}} \tag{3.1}$$

2. We then estimate the shape parameters $\sigma$ using Bayesian approach

3. We then estimate the shape parameters $\alpha$ using Bayesian approach

4. Perform 10,000 repititions of step 1-2

5. We compute the MSE

The results were replicated 10,000 times and the average result were presented in the Tables 1-4. To examine the performance of Bayesian estimates for both shape and shape parameter of Exponentiated Inverse Rayleigh Distribution under different loss functions, the estimate and MSE values obtained by the method of MLE, LLF, ELF and SISLF are shown in Tables 1-4.

**Table 1. Estimates of the parameters of the four methods MLE, LLF, ELF and SISLF with their MLEs**

| n | Parameter | | MLE | | LLF | | ELF | | SISLF | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ |
| 20 | 0.5 | 0.5 | 0.3277 | 0.3037 | 0.5298 | 0.5417 | 0.5280 | 0.5320 | 0.5212 | 0.5195 |
| | a=1.3 | | (0.0708) | (0.0759) | (0.0155) | (0.0269) | (0.0114) | (0.0272) | (0.0105) | (0.0273) |
| | b = 1 | 1.2 | 0.5720 | 1.5060 | 0.5232 | 1.2292 | 0.5204 | 1.3145 | 0.5148 | 1.3401 |
| | k= 2 | | (0.0754) | (0.4735) | (0.0061) | (0.2287) | (0.0060) | (0.2605) | (0.0058) | (0.3204) |
| | | 2 | 0.37725 | 1.0128 | 0.5190 | 2.1340 | 0.516 | 2.2064 | 0.5119 | 2.3326 |
| | | | (0.0561) | (0.3028)` | (0.0047) | (0.8753) | (0.0047) | (0.9440) | (0.0045) | (1.3331) |
| | 1 | 2.5 | 0.8240 | 1.6145 | 1.0464 | 2.8056 | 1.0396 | 2.7898 | 1.0306 | 3.0045 |
| | a=0.5 | | (0.1107) | (0.5340) | (0.0183) | (1.6256) | (0.0178) | (1.7014) | (0.0171) | (2.5415) |
| | b = 1.5 | 1 | 1.1475 | 0.8599 | 1.0582 | 1.1127 | 1.0495 | 1.0791 | 1.0371 | 1.0870 |
| | k= 0.5 | | (0.1800) | (0.2520) | (0.0310) | (0.1654) | (0.0272) | (0.1579) | (0.0257) | (0.1874) |
| | | 1.5 | 1.1328 | 2.1407 | 1.0505 | 1.6725 | 1.0424 | 1.6376 | 1.0327 | 1.6994 |
| | | | (0.1398) | (0.7288) | (0.0226) | (0.4472) | (0.0219) | (0.4476) | (0.0209) | (0.5825) |
| 50 | 0.5 | 0.5 | 0.4793 | 0.4612 | 0.5024 | 0.5033 | 0.5022 | 0.5021 | 0.5015 | 0.50033 |
| | a=1.3 | | (0.0565) | (0.0766) | (0.0015) | (0.0018) | (0.0009) | (0.0018) | (0.0008) | (0.0018) |
| | b = 1 | 1.2 | 0.4947 | 1.1250 | 0.5086 | 1.2271 | 0.5075 | 1.2341 | 0.5051 | 1.2374 |
| | k= 2 | | (0.0452) | (0.2167) | (0.0021) | (0.0060) | (0.0021) | (0.0638) | (0.0021) | (0.0677) |
| | | 2 | 0.4996 | 1.9006 | 0.5073 | 2.0271 | 0.5063 | 2.0660 | 0.5044 | 2.1009 |
| | | | (0.0408) | (0.4087) | (0.0016) | (0.2017) | (0.0017) | (0.2256) | (0.0016) | (0.2544) |
| | 1 | 2.5 | 1.1066 | 2.9527 | 1.0179 | 2.6076 | 1.0151 | 2.5940 | 1.0115 | 2.6571 |
| | a=2.5 | | (0.0817) | (0.6750) | (0.0064) | (0.4002) | (0.0063) | (0.4056) | (0.0062) | (0.4714) |
| | b = 1 | 1 | 1.1063 | 1.2834 | 1.0226 | 1.0421 | 1.0190 | 1.0276 | 1.01401 | 1.0251 |
| | k= 0.5 | | (0.0976) | (0.2524) | (0.0102) | (0.044) | (0.0096) | (0.0427) | (0.0094) | (0.0444) |
| | | 1.5 | 1.0139 | 1.8878 | 1.0204 | 1.5667 | 1.0171 | 1.5494 | 1.0127 | 1.5626 |
| | | | (0.0788) | (0.3849) | (0.0077) | (0.1115) | (0.0076) | (0.1093) | (0.0074) | (0.1189) |

**Table 2. Estimates of the parameters of the four methods MLE, LLF, ELF and SISLF with their MLEs**

| n | Parameter | | MLE | | LLF | | ELF | | SISLF | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ |
| 100 | 0.5 a=1.3 | 0.5 | 0.5862 (0.0453) | 0.5640 (0.067) | 0.5052 (0.0030) | 0.5072 (0.0036) | 0.5049 (0.0018) | 0.5048 (0.0037) | 0.5035 (0.0018) | 0.5014 (0.0037) |
| | b = 1 k= 2 | 1.2 | 0.4559 (0.0281) | 1.2014 (0.1609) | 0.5041 (0.0010) | 1.2129 (0.0281) | 0.5035 (0.0010) | 1.2159 (0.0291) | 0.5023 (0.0010) | 1.2166 (0.0298) |
| | | 2 | 0.4586 (0.0253) | 1.9313 (0.2830) | 0.5034 (0.0008) | 2.0102 (0.094) | 0.5029 (0.0008) | 2.0303 (0.1003) | 0.5020 (0.0008) | 2.0457 (0.1061) |
| | 1 a=2.5 | 2.5 | 1.0481 (0.0553) | 2.3789 (0.3622) | 1.0081 (0.0030) | 2.5512 (0.1775) | 1.0067 (0.0030) | 2.5431 (0.1781) | 1.0048 (0.0029) | 2.5716 (0.1917) |
| | b = 1 k= 0.5 | 1 | 1.1096 (0.0714) | 1.0723 (0.1424) | 1.0108 (0.0045) | 1.0205 (0.0197) | 1.0090 (0.0044) | 1.1096 (0.0192) | 1.0065 (0.0044) | 1.0117 (0.0196) |
| | | 1.5 | 0.9254 (0.0566) | 1.2725 (0.1728) | 1.0112 (0.0038) | 1.5332 (0.0498) | 1.0095 (0.0037) | 1.5242 (0.0491) | 1.0073 (0.0036) | 1.5295 (0.0511) |
| 200 | 0.5 a=1.3 | 0.5 | 0.4926 (0.0285) | 0.4769 (0.0393) | 0.5024 (0.0017) | 0.5038 (0.0018) | 0.5024 (0.0009) | 0.5023 (0.0018) | 0.5017 (0.0008) | 0.5001 (0.0018) |
| | b = 1 k= 2 | 1.2 | 0.5350 (0.0230) | 1.3687 (0.1349) | 0.5018 (0.0005) | 1.2051 (0.0135) | 0.5015 (0.0005) | 1.2065 (0.0137) | 0.5009 (0.0005) | 1.2066 (0.0139) |
| | | 2 | 0.5323 (0.0205) | 2.3485 (0.2590) | 0.5015 (0.0004) | 2.0024 (0.0456) | 0.5013 (0.0003) | 2.0124 (0.0471) | 0.5001 (0.0004) | 2.0196 (0.0484) |
| | 1 a=2.5 | 2.5 | 1.0183 (0.0374) | 2.7745 (0.3128) | 1.0041 (0.0015) | 2.5210 (0.0818) | 1.0034 (0.0015) | 2.5172 (0.0819) | 1.0025 (0.0015) | 2.5296 (0.0845) |
| | b = 1 k= 0.5 | 1 | 1.053 (0.0474) | 1.0707 (0.0995) | 1.0054 (0.0022) | 1.0093 (0.0092) | 1.0044 (0.0021) | 1.0056 (0.0091) | 1.0032 (0.0022) | 1.0047 (0.0091) |
| | | 1.5 | 1.0161 (0.0416) | 1.6512 (0.1681) | 1.0050 (0.0018) | 1.5162 (0.0242) | 1.0042 (0.0018) | 1.5117 (0.0240) | 1.0031 (0.0018) | 1.5140 (0.0244) |

## 3.2 Application to Coating weight by chemical method on Tcs and Bcs.

In this section, the EIRD is applied to two (2) real data sets which were gotten from [15]. The first data set was a 72 observations on coating weight by chemical method on top center side (TCS) and the second data set was 72 observations on coating weight by chemical method on bottom center side (BCS).

For the Tcs data
36.8 47.2 35.6 36.7 55.8 58.7 42.3 37.8 55.4 45.2 31.8 48.3 45.3 48.5 52.8 45.4 49.8 48.2 54.5 50.1 48.4 44.2 41.2 47.2 39.1 40.7 40.3 41.2 30.4 42.8 38.9 34.0 33.2 56.8 52.6 40.5 40.6 45.8 58.9 28.7 37.3 36.8 40.2 58.2 59.2 42.8 46.3 61.2 58.4 38.5 34.2 41.3 42.6 43.1 42.3 54.2 44.9 42.8 47.1 38.9 42.8 29.4 32.7 40.1 33.2 31.6 36.2 33.6 32.9 34.5 33.7 39.9

For the Bcs
45.5 37.5 44.3 43.6 47.1 52.9 53.6 42.9 40.6 34.1 42.6 38.9 35.2 40.8 41.8 49.3 38.2 48.2 44.0 30.4 62.3 39.5 39.6 32.8 48.1 56.0 47.9 39.6 44.0 30.9 36.6 40.2 50.3 34.3 54.6 52.7 44.2 38.9 31.5 39.6 43.9 41.8 42.8 33.8 40.2 41.8 39.6 24.8 28.9 54.1 44.1 52.7 51.5 54.2 53.1 43.9 40.8 55.9 57.2 58.9 40.8 44.7 52.4 43.8 44.2 40.7 44.0 46.3 41.9 43.6 44.9 53.6 The data is summarized in Table 3.

**Table 3. Estimates of the parameters of the four methods MLE, LLF, ELF and SISLF with their MLEs for the real life datasets**

a =3, b = 4 and k = 2

| Data | MLE | | LLF | | ELF | | SISLF | |
|---|---|---|---|---|---|---|---|---|
| | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ | $\sigma$ | $\alpha$ |
| Tcs | 73.0125 | 13.1818 | 71.3211 | 12.2881 | 63.1867 | 12.9297 | 71.3211 | 12.9014 |
| Bcs | 78.5689 | 12.2331 | 76.9089 | 17.0602 | 69.1793 | 17.8291 | 70.1575 | 17.8466 |

# 4 Conclusion

In this work, we consider the classical method and Bayesian method under different loss functions such as Entropy Loss Function, Linex Loss Function and Scale Invariant Squared Error Loss Function. We employed the Bayesian techniques to obtain the posterior estimates of an EIRD using both conjugate and non-conjugate prior distribution under different loss functions and adopted the maximum likelihood approach to estimate the two parameter of interest. Fig. 1. shows that the PDF of the EIRD distribution at varying parameter values which shows that the distribution is positively skewed and the Fig. 2. is the CDF which shows the increasing pattern as other distributions. Fig. 3. shows the reliability graph which proves that the distribution can be used in lifetime studies since the graph tends to decrease as the time increases. Fig. 4. shows the hazard graph which shows the upside down bath-tub curve shape.

Table 1 and 2 shows the posterior estimates with MSE under different loss functions for the simulated datasets. Table 3, shows the posterior estimate on the real life dataset (coating weight by chemical method on top center side (TCS) and bottom center side (BCS))for different prior distribution under different loss functions

Based on the results displayed in Tables 1 and 2, we observed that all the posterior estimates for both shape and scale parameters for the simulated datasets are close to the true values of parameters of an EIRD. Also, we discovered the methods are consistent since the values of MSE decrease as sample size increases. It can be observed that the Bayesian estimates for both scale and shape parameters under the Bayesian techniques perform better than that of the classical techniques . The results obtained under the loss function ELF were quite more efficient than others loss functions because of its smallest MSE.

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# Competing Interests

Authors have declared that no competing interests exist.

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