Asian Research Journal of Mathematics

17(2): 102-110, 2021; Article no.ARJOM.67127

ISSN: 2456-477X

On the Irrationality and Transcendence of Rational Powers of e

Sourangshu Ghosh[1*]

[1] *Department of Civil Engineering, Indian Institute of Technology Kharagpur, India.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2021/v17i230277

Editor(s):

(1) Dr. Sakti Pada Barik, Gobardanga Hindu College, India.

Reviewers:

(1) Gilbert Makanda, Central University of Technology, South Africa.

(2) K. S. Vidhyaa, Anna Univeristy, India.

Complete Peer review History: http://www.sdiarticle4.com/review-history/67127

Received: 27 February 2021

Accepted: 02 April 2021

Published: 27 April 2021

Review Article

Abstract

A number that can't be expressed as the ratio of two integers is called an irrational number. Euler and Lambert were the first mathematicians to prove the irrationality and transcendence of e. Since then there have been many other proofs of irrationality and transcendence of e and generalizations of that proof to rational powers of e. In this article we review various proofs of irrationality and transcendence of rational powers of e founded by mathematicians over the time.

Keywords: Irrationality; transcendence; Euler's number.

2020 Mathematics Subject Classification: 11J72, 11J81.

1 Background

The most well known proof of Irrationality of e was proven by Joseph Fourier using proof by contradiction [1]. Before that Euler already wrote the first proof of Irrationality of e using the simple continued fraction expansion of e back in 1737 [2]-[4].

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\cdots}}}}}} \tag{1.1}$$

*Corresponding author: E-mail: sourangshug123@gmail.com;

This is an infinite simple continued fraction, which is always irrational. A more simpler proof of this continued fraction was given by Cohn [5]. The proof by contradiction given by Fourier works like this:

Let us assume that e is a rational number and can be expressed as $\frac{p}{q}$, where p, q are integers. Now e can be expressed as:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \sum_{n=0}^{q} \frac{1}{n!} + \sum_{n=q+1}^{\infty} \frac{1}{n!}$$

Multiplying both sides by $q!$,we get

$$p(q-1)! = \sum_{n=0}^{q} \frac{q!}{n!} + \sum_{n=q+1}^{\infty} \frac{q!}{n!}$$

Both the LHS and first term of RHS are integer, but the second term is

$$\sum_{n=q+1}^{\infty} \frac{q!}{n!} < \sum_{n=1}^{\infty} \frac{1}{(q+1)^n} = \frac{1}{(q+1)} \frac{1}{(1-\frac{1}{(q+1)})} = \frac{1}{q} < 1 \tag{1.2}$$

which is not an integer. Hence we arrive at a contradiction. MacDivitt [6] gave a proof similar to the above proof, it uses the fact that

$$(b+1)x = 1 + \frac{1}{b+2} + \frac{1}{(b+2)(b+3)} + ... < 1 + \frac{1}{(b+1)} + \frac{1}{(b+1)(b+2)} + ... = 1 + x \tag{1.3}$$

which proves that $bx < 1$, but that is not possible since both b and x are integers.

Penesi [7], Apostol [8] proved this by proving e^{-1} instead of proving e irrational. Note that the expansion of e^{-1} is

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

Let us define the truncated part of the expansion as $t_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ Therefore we can write

$$e^{-1} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} + \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}$$

Let us assume $e^{-1} = \frac{m}{n}$. Multiplying both sides of the previous equation by $n!$, we get the LHS as an integer, and the first term of the RHS as an integer. Therefore we must have

$$n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}$$

as an integer.But this also satisfies

$$0 \le \left| n! \sum_{n+1}^{\infty} \frac{(-1)^k}{k!} \right| \le \frac{n!}{(n+1)!} \le 1. \tag{1.4}$$

We therefore arrive at a contradiction and hence e^{-1} is irrational.Higher powers of e were subsequently also proven to be irrational. The irrationality of e^2 was proven in [9] , of e^3 in [10], and of e^4 in [11].

2 Proof using Niven's Polynomials

A more generalized result where the power is a rational number was proven by Niven in 1985. It is first proved by Ivan Niven [12] that $e^{x/y}$ is an irrational number using Niven's Polynomials of the form $\frac{x^n(1-x)^n}{n!}$, which can be also be used to prove that π is an irrational number. A similar proof was also given by Aigner [13], Beatty [14] and Eugeni [15].

Let us define $f : [0,1] \longrightarrow \mathbb{R}$, $f(x) = \dfrac{x^n(1-x)^n}{n!}$ then we have $f(x) = f(1-x)$ and $0 \le f(x) < \dfrac{1}{n!}$. We also note that these functions satisfy

$$f^{(j)}(0)\ ,\ f^{(j)}(1) \in \mathbb{Z}\ ,\ j \ge 0$$

Let us assume that $e^p = \dfrac{a}{b}$, where p is an integer. Let us define another function F as

$$F = p^{2n}f - p^{2n-1}f' + p^{2n-2}f'' - \cdots + f^{(2n)} \tag{2.1}$$

This function satisfies

$$F' + pF = p^{2n+1}f$$

Multiplying both sides by be^{px} and then integrating we get

$$b\Big[e^{px}F(x)\Big]_0^1 = b\int_0^1 p^{2n+1}e^{px}f(x)dx \longrightarrow 0^+$$

as $n \longrightarrow \infty$. Now note that the LHS is $b[e^pF(1) - F(0)] = aF(1) - bF(0)$ which must belong to $\mathbb{Z}^+$. But that is not possible, therefore we arrive at a contradiction.Now as e^p is an integer, any root of that number $(e^p)^{\frac{1}{q}}$ will also be an irrational number. Another beautiful proof using polynomials of similar form was stated by Joe Mercer [16]. Let us take the two integrals:

$$I_n = \frac{1}{n!}\int_0^\infty [x(x-p)]^n e^{-x}dx, J_n = \frac{1}{n!}\int_0^\infty [x(x+p)]^n e^{-x}dx \tag{2.2}$$

As the polynomials inside the integral (2.2) $[x(x-p)]^n$,$[x(x+p)]^n$ have integer coefficients and the least power of x is n, we must have both I_n and J_n as integer as $\int_0^\infty x^k e^{-x}dx = k!$. Let us assume that $e^p = \frac{f}{g}$. Let us multiply e^p by gI_n.We then have

$$\begin{aligned} ge^pI_n &= \frac{ge^p}{n!}\int_0^\infty [x(x-p)]^n e^{-x}dx + \frac{g}{n!}\int_0^\infty [x(x-p)]^n e^{-(x-p)}dx \\ &= \frac{ge^p}{n!}\int_0^\infty [x(x-p)]^n e^{-x}dx + \frac{g}{n!}\int_0^\infty [u(u+p)]^n e^{-u}du \\ &= \frac{ge^p}{n!}\int_0^\infty [x(x-p)]^n e^{-x}dx + gJ_n \end{aligned}$$

Now note that since $x|x-p| \le \frac{p^2}{4}$ in $[0,p]$ and $0 < e^{-x} \le 1$, we have

$$|\frac{ge^p}{n!}\int_0^\infty [x(x-p)]^n e^{-x}dx| \le \frac{me^pp^{2n}}{4^n n!} \tag{2.3}$$

Now since factorial grow faster than exponential, we can choose an n such that $n! > me^p(\frac{p^2}{4})^n$.Also note that $\int_0^\infty [x(x-p)]^n e^{-x}dx$ will be never be 0 since $[x(x-p)]^n e^{-x}$ never changes in sign in $[0,p]$. Thus we have due to (2.3)

$$ge^pI_n = \epsilon_n + gJ_n$$

Here $0 < |\epsilon_n| < 1$ if $n! > me^p(\frac{p^2}{4})^n$. We therefore arrive at a contradiction since RHS of the above mentioned equation can't be an integer. We hence proved that there is no integral multiple of e^p which can be an integer.

3 Proof using Continued Fractions

The proof stated in the section is discussed by Ghosh [17, 18].We start with the Continued Fraction Expansion of the hyperbolic tanh function discovered by Gauss [19, 20]

$$\tanh z = \cfrac{z}{1 + \cfrac{z^2}{3 + \cfrac{z^2}{5 + \cfrac{z^2}{\cdots}}}}.$$

We also know that the hyperbolic tanh function is related to the exponential function with the following formula

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

Putting $\frac{x}{y}$ in place of z in the previous equation we get

$$\frac{e^{\frac{x}{y}} - e^{-\frac{x}{y}}}{e^{\frac{x}{y}} + e^{-\frac{x}{y}}} = \cfrac{(\frac{x}{y})}{1 + \cfrac{(\frac{x}{y})^2}{3 + \cfrac{(\frac{x}{y})^2}{5 + \cfrac{(\frac{x}{y})^2}{\cdots}}}}.$$

This continued fraction can be simplified into

$$\frac{e^{\frac{x}{y}} - e^{-\frac{x}{y}}}{e^{\frac{x}{y}} + e^{-\frac{x}{y}}} = \cfrac{x}{y + \cfrac{x^2}{3y + \cfrac{x^2}{5y + \cfrac{x^2}{\cdots}}}}.$$

This equation can be further be simplified as

$$1 + \frac{2}{e^{\frac{2x}{y}} - 1} = y + \cfrac{x^2}{3y + \cfrac{x^2}{5y + \cfrac{x^2}{\cdots}}}$$

$$\frac{e^{\frac{2x}{y}} - 1}{2} = \cfrac{1}{(\frac{y}{x} - 1) + \cfrac{x}{3y + \cfrac{x^2}{5y + \cfrac{x^2}{\cdots}}}}.$$

Some algebraic manipulation, yields a continued fraction expansion of $e^{x/y}$

$$e^{x/y} = 1 + \cfrac{2x}{2y - x + \cfrac{x^2}{6y + \cfrac{x^2}{10y + \cfrac{x^2}{14y + \cfrac{x^2}{18y + \ddots}}}}},$$

which is an infinite continued fraction. Legendre found necessary and sufficient conditions for the convergence of the continued fraction in following theorem. The conditions were first published by Chrystal [21].

Theorem 3.1. *The necessary and sufficient condition that the continued fraction*

$$\cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}$$

is irrational is that the values a_i, b_i are all positive integers, and there is an integer n such that $|a_i| > |b_i|$ for all i greater than n.

In the continued fraction of $e^{x/y}-1$, we have derived a_i, b_i are equals to $2(2i-1), x^2$ except when $i=1$. Therefore we have $|a_i| > |b_i|$ for all $i > \frac{\frac{x^2}{2}+1}{2}$. Hence we have proved that $e^{x/y}-1$ is irrational which in turn means $e^{x/y}$ is irrational, where x, y are integers.

4 Transcendence of Rational Powers of e

A transcendental number is a number that cannot be expressed as the root of a non-zero polynomial with all its coefficients being rational. Note that an irrational number does not necessary have to be a transcendental number. Square-root of any non-square integer is a irrational number but not a transcendental number. A number that is not transcendental is called an algebraic number.

It was first Hermite [22, 23] who proved that e is transcendental. This results were further extended by Lindemann who proved that e^{α} is transcendental, given α is a non-zero transcendental number [24, 25]. Using this he also proved that π is transcendental, since $e^{i\pi} = -1$, which is a real number. Weierstrass generalized this proof [26] to give the well known LindemannWeierstrass theorem. Hilbert [27], Gordan [28] simplified this proof. A similar theorem establishing that a^b is a transcendental number given that a is an algebraic number satisfying $a \neq 0, 1$ and b is an algebraic number which is irrational but not transcendental was proved by Gelfond [29] known as GelfondSchneider theorem. This two theorem were further extended by Baker [30]. All of these theorems is generalized further by Schanuel's conjecture [31]. Bernard [32] proved the transcendence of e using multivariate and symmetric Polynomials.

In this article we shall only discuss about the transcedence of of rational powers of e. To prove that e^v is transcendental, where v is a rational number, let is assume that e^v is algebraic and satisfies

$$c_0 + c_1e^v + c_2e^{2v} + ... + c_ne^{nv} = 0 \tag{4.1}$$

where all coefficients $c_t(0 \leq t \leq n)$ are integers with c_0, c_n being non-zero. We now employ a function which is an extension of Niven's Polynomials:

$$f_k(x) = v^{2k+2}x^k[(x-1)...(x-n)]^{k+1}$$

Note that the least power of x in $f_k(x)$ is k, but the least power of x in $f_k(x+a)$, where a is $0 < a \leq n$ is $k+1$. Multiplying both sides of (4.1) by $\int_0^\infty f_k e^{-vx}dx$, we get the following equation:

$$\sum_{t=0}^{n} c_t e^{tv}\left(\int_0^\infty f_k e^{-vx}dx\right) = 0$$

The LHS can be divided into two parts P, Q such that $P + Q = 0$

$$P = \sum_{t=0}^{n} c_t e^{tv}\left(\int_t^\infty f_k e^{-vx}dx\right) \tag{4.2}$$

$$Q = \sum_{t=1}^{n} c_t e^{tv}\left(\int_0^t f_k e^{-vx}dx\right) \tag{4.3}$$

We now derive two lemmas to prove the transcendence of e.

Lemma 4.1. $\frac{P}{k!}$ *is a positive integer*

Note that every term in P will contain sum of integer multiples of integrals of the form

$$\int_0^\infty x^j e^{-vx}dx = \frac{j!}{v^{j+1}} \tag{4.4}$$

which is the value of gamma function at integer points.Note that the integrand $c_t e^{tv} \int_t^{\infty} f_k e^{-vx} dx$ for every t satisfying $0 < t \leq n$ is a sum of terms whose lowest and highest power of x is $k+1, 2k+1$ respectively, multiplied with e^{-vx} integrated from 0 to ∞ after substituting x for $x+a$ since

$$c_t e^{tv} \int_t^{\infty} f_k e^{-vx} dx = c_t e^{tv} \int_0^{\infty} f_k(x+t) e^{-v(x+t)} dx = c_t \int_0^{\infty} f_k(x+t) e^{-vx} dx$$

Therefore P can be written as

$$P = c_0 e^0 (\int_0^{\infty} f_k e^{-vx} dx) + \sum_{t=1}^{n} c_t \sum_{j=k+1}^{2k+1} A_{j-k,t} v^{2k-j+1} (v^{j+1} \int_0^{\infty} x^j e^{-vx} dx) \tag{4.5}$$

Substituting (4.4) in (4.5), we get

$$P = c_0 e^0 (\int_0^{\infty} f_k e^{-vx} dx) + \sum_{t=1}^{n} c_t \sum_{j=k+1}^{2k+1} A_{j-k,t} v^{2k-j+1} j!$$

Here $A_{j-k,t}$ refers to the integer coefficient of x^j in $\frac{f_k(x+t)}{v^{2k+2}}$. All the terms in the second part of RHS of (4.5) are divisible by $(k+1)!$. Therefore after division by $k!$, it must be also divisible by $(k+1)$. The first part of RHS of (4.5) can be expressed as

$$c_0 e^0 (\int_0^{\infty} f_k e^{-x} dx) = \int_0^{\infty} v^{2k+2} ([(-1)^n (n!)]^{k+1}) e^{-vx} x^k + ...) dx$$

The higher order terms in RHS shall be divisible by $(k+1)$. Therefore we get

$$\frac{1}{k!} c_0 (\int_0^{\infty} f_k e^{-vx} dx) \equiv c_0 [(-1)^n (n!)]^{k+1}) v^{k+1} \not\equiv 0 (\text{mod} k+1) \tag{4.6}$$

We see that $\frac{P}{k!}$ is not divisible by $k+1$, if it is a prime greater than $n, |c_0|$. But since P is divisible by $k!$, $\frac{P}{k!}$ cannot be zero.

Lemma 4.2. *There exists some k such that $|\frac{Q}{k!}| < 1$*

Let us start with two continuous functions $g(x), f(x)$

$$g(x) = v^2 x(x-1)...(x-n) \tag{4.7}$$

$$f(x) = v^2 (x-1)...(x-n) e^{-vx} \tag{4.8}$$

Since both of them are continuous functions, they are bounded in the interval $[0, n]$. Let the upper bounds be $b_1, b_2 > 0$ respectively. Therefore $f_k e^{-vx} = g(x)^k f(x)$ is also bounded by $b_1^k b_2$ in the interval $[0, n]$. Each of the integrals are themselves bounded since

$$|\int_t^n f_k e^{-vx} dx| \leq \int_t^n |f_k e^{-vx}| dx \leq \int_t^n b_1^k b_2 dx = (n-t) b_1^k b_2$$

Therefore the sum Q is itself bounded as

$$|Q| < n b_1^k b_2 (c_0 + c_1 e^v + c_2 e^{2v} + ... + c_n e^{nv}) = b_1^k w \tag{4.9}$$

Here $w = n b_2 (c_0 + c_1 e^v + c_2 e^{2v} + ... + c_n e^{nv})$ is independent of k. Therefore we get

$$|\frac{Q}{k!}| < w \frac{b_1^k}{k!} \to 0 \;\; as \;\; k \to \infty$$

Theorem 4.3. *e^v is a transcendental number.*

We note that

$$\frac{1}{k!}\sum_{t=0}^{n} c_t e^{tv}(\int_0^\infty f_k e^{-vx}dx) = \frac{1}{k!}(P+Q) = 0$$

But $\frac{P}{k!}$ is a positive integer whereas $\frac{Q}{k!}$ is a very small real number close to zero. The sum of $\frac{P}{k!}$ and $\frac{Q}{k!}$ can never be zero. Therefore our original assumption is wrong. Hence e^v does not satisfies

$$c_0 + c_1 e^v + c_2 e^{2v} + ... + c_n e^{nv} = 0$$

where all coefficients $c_t(0 \leq t \leq n)$ are integers with c_0, c_n being non-zero.Therefore e^v is transedental number. Since any n^{th} root of e^v is also transcendental number, we must have $e^{p/q}$ a transcendental number for any rational number p/q.

5 Conclusion

In this article we reviewed various proofs of irrationality and transcendence of rational powers of e founded by mathematicians over the time.

Competing Interests

Author has declared that no competing interests exist.

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Peer-review history:
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