Applied Physics Research; Vol. 11, No. 2; 2019
ISSN 1916-9639 E-ISSN 1916-9647
Published by Canadian Center of Science and Education

Newton's Parabola Observed from Pappus' Directrix, Apollonius' Pedal Curve (Line), Newton's Evolute, Leibniz's Subtangent and Subnormal, Castillon's Cardioid, and Ptolemy's Circle (Hodograph) (09.02.2019)

Jiří Stávek[1]

[1] Bazovského, Prague, Czech Republic

Correspondence: Jiří Stávek, Bazovského 1228, 163 00 Prague, Czech Republic. E-mail: stavek.jiri@seznam.cz

Received: February 2, 2019 Accepted: February 20, 2019 Online Published: February 25, 2019

doi:10.5539/apr.v11n2p30 URL: http://dx.doi.org/10.5539/apr.v11n2p30

Abstract

Johannes Kepler and Isaac Newton inspired generations of researchers to study properties of elliptic, hyperbolic, and parabolic paths of planets and other astronomical objects orbiting around the Sun. The books of these two Old Masters "Astronomia Nova" and "Principia…" were originally written in the geometrical language. However, the following generations of researchers translated the geometrical language of these Old Masters into the infinitesimal calculus independently discovered by Newton and Leibniz. In our attempt we will try to return back to the original geometrical language and to present several figures with possible hidden properties of parabolic orbits. For the description of events on parabolic orbits we will employ the interplay of the directrix of parabola discovered by Pappus of Alexandria, the pedal curve with the pedal point in the focus discovered by Apollonius of Perga (The Great Geometer), and the focus occupied by our Sun discovered in several stages by Aristarchus, Copernicus, Kepler and Isaac Newton (The Great Mathematician). We will study properties of this PAN Parabola with the aim to extract some hidden parameters behind that visible parabolic orbit in the Aristotelian World. In the Plato's Realm some curves carrying hidden information might be waiting for our research. One such curve - the evolute of parabola - discovered Newton behind his famous gravitational law. We have used the Castillon's cardioid as the curve describing the tangent velocity of objects on the parabolic orbit. In the PAN Parabola we have newly used six parameters introduced by Gottfried Wilhelm Leibniz - abscissa, ordinate, length of tangent, subtangent, length of normal, and subnormal. We have obtained formulae both for the tangent and normal velocities for objects on the parabolic orbit. We have also obtained the moment of tangent momentum and the moment of normal momentum. Both moments are constant on the whole parabolic orbit and that is why we should not observe the precession of parabolic orbit. We have discovered the Ptolemy's Circle with the diameter a (distance between the vertex of parabola and its focus) where we see both the tangent and normal velocities of orbiting objects. In this case the Ptolemy's Circle plays a role of the hodograph rotating on the parabolic orbit without sliding. In the Plato's Realm some other curves might be hidden and have been waiting for our future research. Have we found the Arriadne's Thread leading out of the Labyrinth or are we still lost in the Labyrinth?

Keywords: Newton's Parabola, Aristotelian World, Plato's Realm, Hidden Mathematical Objects, Pappus' Directrix, Apollonius' Pedal Curves, PAN Parabola, Newton's Evolute, Castillon's Cardioid, Leibniz's Subtangent and Subnormal, Ptolemy's Circle

1. Introduction

The famous quote of Heraclitus "Nature loves to hide" was described in details by Pierre Hadot in 2008. Hadot in his valuable book gives us many examples how Nature protects Her Secrets. In several situations the enormous research of many generations is strongly needed before the right "recipe" unlocking the true reality can be found.

Conic sections - Circle, Ellipse, Hyperbola, and Parabola - are among the oldest curves, and belong to the Treasure of Geometry. The conic sections seem to have been discovered by Menaechmus and were thoroughly studied by Apollonius of Perga (The Great Geometer) and his scholars. The conics are endowed with numerous beautiful properties, some those properties are shared by the entire family, while some properties are unique and original for each of them. In our research we have to be very careful and patient when we want to apply those properties for the planet orbits. Menaechmus said to Alexander the Great: "O King, for travelling over the country, there are royal roads and roads for common citizens, but in geometry there is one road for all."

Parabola is a very original conic section with its own Beauty and Secrets. Though, it has only one focus, it might reveal similar properties as Her Sisters Ellipse and Hyperbola. Pappus of Alexandria discovered the directrix and focus of the parabola, and Apollonius of Perga systematically revealed numerous properties of the parabola. This Ancient Treasure passed into the hands of Copernicus, Galileo, Kepler, Huygens, Newton, Leibniz and many others.

W.R. Hamilton in 1847 discovered how to find the tangent velocity for the elliptic orbit using the auxiliary circle of that ellipse. This technique works very well for hyperbola with two foci, too. But how to find the tangent and normal velocities for parabola with only one focus?

The support came to us from the Great Old Master - Gottfried Wilhelm Leibniz. Gottfried Wilhelm Leibniz during his preparation work on the infinitesimal calculus introduced six parameters for a point on the parabola: abscissa, ordinate, length of tangent, subtangent, length of normal, and subnormal. For his infinitesimal triangle Leibniz used the ratio of ordinate to subtangent and the ratio of subnormal to ordinate. Another important inspiration came to us from Johann Castillon with his cardioid - the inversion curve to parabola. This curve might bring an information about the tangent velocity of an object on parabolic orbits.

Can we try to employ the Castillon - Leibniz concept in order to get the needed tangent and normal velocities for objects on parabolic orbit? Can we find a hodograph circle around the PAN Parabola? Who can support us in this research? How about the Great Old Master - Claudius Ptolemy and his Circles?

A possible chance for further classical development of the Newton's parabola is to penetrate more deeply into the secrets of the Newton's parabola and to reappear with some new hidden properties overlooked by earlier generations of researchers. Our guiding principle is the existence of the Plato's Realm with invisible mathematical objects that might bring to us some additional information about the visible Newton's parabola in the Aristotelian World. In this contribution we have been working with these mathematical objects from the Plato's Realm:

1. Parabola properties discovered by Apollonius of Perga - the Great Geometer - and many his scholars.

2. Directrix and focus of parabola discovered by Pappus of Alexandria

3. Locus of the radii of curvature (evolute) of parabola - Isaac Newton in 1687.

4. The interplay of the directrix of parabola P (= Pappus - the discoverer of directrix), the vertex of the parabola A (= Apollonius of Perga - the Great Geometer), and the occupied focus N (= Isaac Newton - the Great Mathematician) together forms the PAN Parabola with several interesting hidden properties of those parabolic paths.

5. Castillon's cardioid (the inverse curve to parabola) as the measure of tangent velocities on parabolic orbits.

6. Pedal curve with the pedal point in the Newton's focus - the "auxiliary circle" - the line perpendicular to the axis of parabola in the vertex.

7. Tangent and normal to a point on the parabolic orbit described by six parameters introduced by Gottfried Wilhelm Leibniz: abscissa, ordinate, length of tangent, subtangent, length of normal, and subnormal.

8. Moment of tangent momentum, moment of normal momentum.

9. Hodograph: Ptolemy's Circle with the diameter a (distance between the vertex of parabola and its focus) where we see both the tangent and normal velocities of orbiting object. In this case the Ptolemy's Circle plays a role of the hodograph rotating on the parabolic orbit without sliding.

The experimental analysis of properties of the PAN Parabola should reveal if we have found the Arriadne's Thread leading out of the Labyrinth or if we are still lost in the Labyrinth.

(We are aware of the famous quote of Richard Feynman from the year 1962: "There's certain irrationality to any work in gravitation, so it is hard to explain why you do any of it.")

2. Some Properties of the PAN Parabola

Figure 1, Figure 2, and Figure 3 show some parabolic properties of the PAN parabola that might be used for the description of motion of objects around the Sun on the parabolic orbits. Some of those parameters are very well-known from many good books on conic sections, some parameters are newly derived. The great inspiration was found in the works of Isaac Todhunter (1881), who studied very deeply the properties of conic sections.

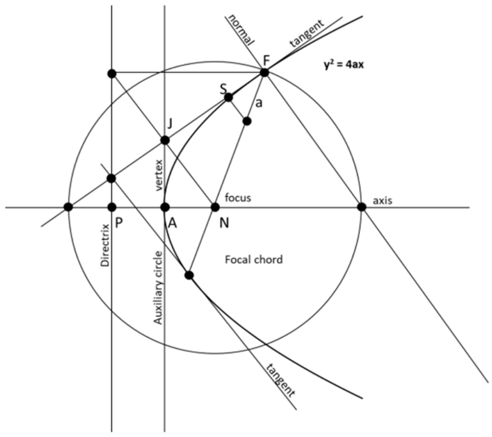

Figure 1. Some properties of the PAN Parabola: Directrix P (=Pappus), vertex A (=Apollonius), occupied focus N (=Newton), tangent, normal, focal chord

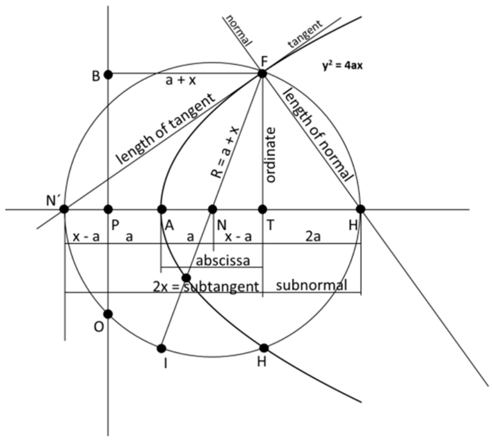

Figure 2. Some properties of the PAN Parabola with PANTHEION (Πάνθειον) Circle, and six Leibniz's parameters: Abscissa, ordinate, length of tangent, subtangent, length of normal, subnormal for the point F on the parabolic orbit

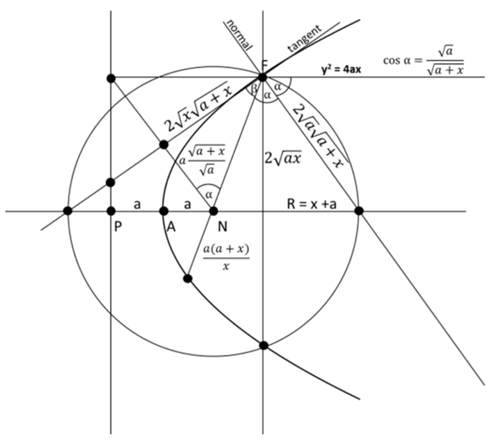

Figure 3. Some properties of the PAN Parabola for the quantitative calculation

Table 1 summarizes some relations for the PAN parabola.

Table 1. Some properties of the PAN parabola

Some properties of the PAN parabola $y^2 = 4\,ax$
a = 1 = PA = AN … parameter of the PAN parabola
Latus rectum = 4a
PANTHEION' Circle for the construction of the tangent and the normal to a point F (Πάνθειον)
P … Pappus' directrix
A … Apollonius' vertex
N … Newton's occupied focus
F … object on the parabolic orbit
J, S … points on the tangent
ρ … radius of curvature of hyperbola
Leibniz's six parameters of a point on the parabolic orbit
$Abscissa \quad x$
$Ordinate \quad y = 2\sqrt{ax}$
$Length\,of\,the\tan gent = 2\sqrt{x}\sqrt{a+x}$
$Sub\tan gent = 2x$
$Length\,of\,the\,normal = 2\sqrt{a}\sqrt{a+x}$
$Subnormal = 2a$
$NF = BF = NH = a + x$
SURF = Ptolemy's Circle
$JN = a\dfrac{\sqrt{a+x}}{\sqrt{a}}$
$SU = a\dfrac{\sqrt{a}}{\sqrt{a+x}}$
$SF = a\dfrac{\sqrt{x}}{\sqrt{a+x}}$
$\rho = \dfrac{2\left(a+x\right)^{3/2}}{\sqrt{a}}$

3. Proposed Reflecting and Refracting Properties of Solar and Object Gravitons on the Parabolic Orbit

In this section we will assume that Solar gravitons enter into the internal volume of objects on the parabolic orbits and collide with object gravitons in four possible scenarios as it was in details described by Jiří Stávek (2018). Gravitons are reflected and refracted similarly as in the case of the Kepler's ellipse, and Newton's Hyperbola. Therefore, we will present here only a qualitative schema of reflective and refractive properties of parabola in Figure 4.

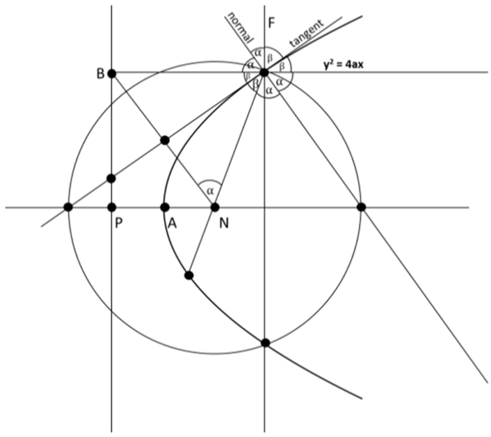

Figure 4. Reflective and refractive properties of parabola

4. Castillon's Cardioid as the Remarkable Curve for the Determination of Tangent Velocities of Objects on Parabolic Orbits

During our tour through the Plato's Realm we were studying the properties of the cardioid - from the Greek καρδία "heart", "Herzkurve" - as the inverse curve to parabola. Cardioid was intensively studied by many researchers: Römer (1674), Vaumesle (1678), Koërsma (1689), Ozanam (1691). The name cardioid coined Johann Castillon in 1741. Laura Pennington (her web site was found on 28.01.2019) observed the formation of the cardioid in the cross section of an Apple Cut in Half. **How to cut the Newton's Apple in order to get the tangent velocity of objects on the parabolic orbit?**

We have found this remarkable formula for the tangent velocity v_T within several minutes:

$$v_T^2 = v_C^2(1 - \cos\varphi) = v_C^2\left(1 - \frac{x-a}{x+a}\right) = v_C^2\frac{2a}{a+x} = v_{ESC}^2\frac{a}{a+x} \tag{1}$$

where v_T is the tangent velocity, v_C is the circular velocity, φ is the angle between the axis of parabola and the line connecting the Newton's focus with the object on parabolic orbit, v_{ESC} is the very well-known escape velocity for the parabolic orbit.

The great inspiration for this concept was delivered to us by the book of Viktor Blåsjö (2017) "Transcendental Curves in the Leibnizian Calculus" where author describes in details the works of Old Masters before the infinitesimal calculus started to dominate the book "Principia…".

5. Newton's Parabola Observed from the Newton's Evolute

Newton discovered several important properties hidden in the Kepler's ellipse, the Newton's Hyperbola and the Newton's Parabola in his Principia in 1687. For the centripetal force F, he derived formula:

$$F = m\frac{v_T^2}{\rho} \tag{2}$$

where m is the mass of the planet (or any object), v_T is the tangent velocity of the planet and ρ is the radius of curvature of that ellipse, hyperbola or parabola. The locus of radii of curvature is termed as the evolute. This equation opened completely new possibilities in the understanding of the Kepler's ellipse, Newton's Hyperbola, and Newton's Parabola. We will use this procedure for the description of events in Newton's parabola.

In the standard procedure both quantities v_T and ρ are found by the infinitesimal calculus discovered independently by Newton and Leibniz.

We will present here the trigonometric approach to these two quantities (v_T in the next chapter). The radius of curvature of the hyperbola ρ can be derived in the trigonometric way shown in Figure 5. Figure 5 describes an interplay between the normal to the tangent and the line connecting the Sun and orbiting planet.

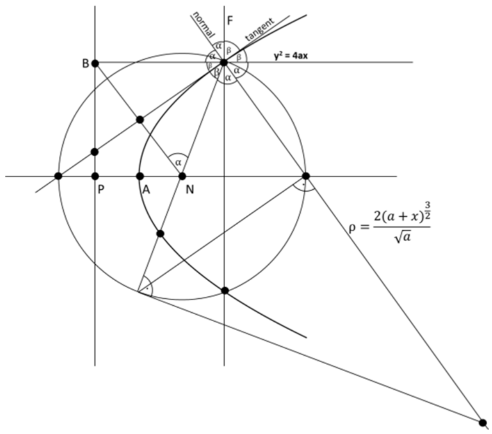

Figure 5. Trigonometric approach to reveal the expression for the radius of curvature ρ of the Newton's parabola

We have used the deep knowledge of parabola properties of Issac Todhunter (1881) and extracted from Figure 5 the expression for the radius of curvature, which is already known in the existing literature:

$$\rho = \frac{2\left(a+x\right)^{3/2}}{\sqrt{a}} \quad (3)$$

(The quantities expressed in the trigonometric language are simpler and Nature can talk with us in this trigonometric language that could be depicted in simple Figures without words. The trigonometric function cosα is cleverly hidden in the Equation 3).

6. Newton's Parabola Observed from the Pedal Curve with Pedal Point in the Newton's Focus and from the Leibniz's Subnormal

The pedal curve of the Newton's parabola is the locus of the feet of the perpendiculars from the occupied focus to the tangent of that parabola. In this case the pedal curve is the famous "auxiliary circle" of the parabola - the line perpendicular to the axis of parabola in the vertex of that parabola. The distance JN can be used for the arm of the moment of tangent momentum - see Figure 6.

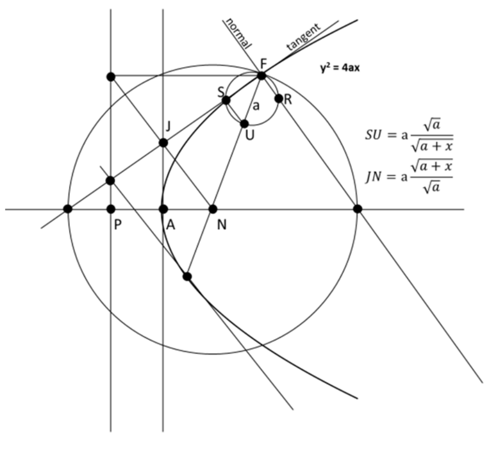

Figure 6. PAN Parabola with the distance JN (length of the arm for the moment of tangent momentum) and with the SURF Circle (Ptolemy's Circle - the Hodograph) where SU represents the tangent velocity on the parabolic orbit

We were inspired by W.R. Hamilton who in 1847 discovered his concept for the Kepler's ellipse that is known as the hodograph. This approach was several times forgotten and its Beauty was several times rediscovered by many researchers. E.g., Richard Feynman in his "Lost lecture" made this concept very well known for our generation. However, this great idea does not work for parabola - there is no empty focus of parabola.

Luckily, we were inspired by another Great Old Master - Gottfried Wilhelm Leibniz. Gottfried Wilhelm Leibniz during his preparation work on the infinitesimal calculus introduced six parameters for a point on the parabola: abscissa, ordinate, length of tangent, subtangent, length of normal, and subnormal. Leibniz studied the change of these six parameters and their dependence on the direction of tangent and normal and their dependence on the curvature in that point on the parabola. For his infinitesimal triangle Leibniz used the ratio of ordinate to subtangent and the ratio of subnormal to ordinate. Can we extract any additional information hidden in those six Leibniz's parameters? For details see D. Dennis (1995), D. Dennis and J. Confrey (1995). [See also A. R. Hall (2015) and T. Sonar (2018)].

For the tangent velocity v_T of an object on the parabolic orbit we propose to use the ratio of the subnormal to the length of the normal:

$$v_T = v_c \frac{\sqrt{2}\sqrt{a}}{\sqrt{a+x}} = v_{ESC} \frac{\sqrt{a}}{\sqrt{a+x}} = v_{ESC} \frac{2a}{2\sqrt{a}\sqrt{a+x}} \tag{4}$$

where v_{ESC} is the very well-known escape velocity just to leave the Sun or the planet. The first part of this Equation was given to us by Castillon from his cardioid, the second part of that Equation was presented to us by G.W. Leibniz.

Now, we can test the validity of the Newton's formula expressed in the trigonometric language and compare these trigonometric formulae with formulae obtained in other mathematical languages.

The famous Newton's formula can be trigonometrically expressed as:

$$F = \frac{GMm}{R^2} = m \frac{v_T^2}{\rho} \frac{1}{\cos\alpha} = m \frac{v_c^2 \left(\frac{2a}{a+x}\right)}{\frac{2(a+x)^{3/2}}{\sqrt{a}}} \frac{\sqrt{a+x}}{\sqrt{a}} = \frac{m\, v_c^2 a}{R^2} \tag{5}$$

We have inserted the expression for the escape velocity $v_{ESC} = 2^{1/2}\, v_C$, where v_C is the velocity for the circular orbit. At the end we have obtained the standard gravitational parameter μ:

$$\mu = G\, M = a\, v_C^2 \tag{6}$$

where a is the parameter of the PAN Parabola and v_C is the circular orbital velocity for that object.

7. Newton's Parabola Observed from the Contrapedal Curve with Pedal Point in the Newton's Focus and from the Leibniz's Subtangent

We want to find an expression for the normal velocity v_N of an object on the parabolic orbit. In this case the PAN Parabola shows Her original Feature and Beauty. We will use the contrapedal distance NC for the arm of the moment of the normal momentum from Figure 7.

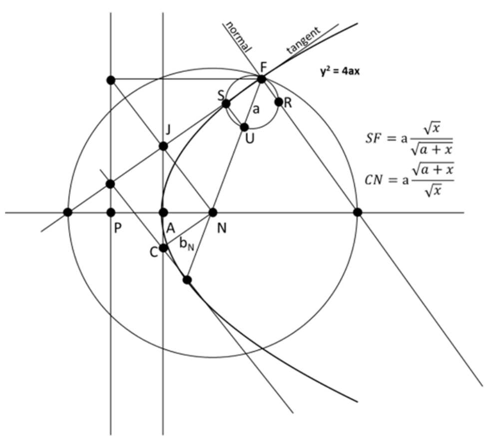

Figure 7. PAN Parabola with the distance NC (length of the arm for the moment of normal momentum) and with the SURF Circle (Ptolemy's Circle - the Hodograph) where SF represents the normal velocity on the parabolic orbit

For the normal velocity v_N of an object on the parabolic orbit we propose to use the ratio of the subtangent to the length of the tangent:

$$v_N = v_{esc} \frac{2x}{2\sqrt{x}\sqrt{a+x}} = v_{esc} \frac{\sqrt{x}}{\sqrt{a+x}} \tag{7}$$

where v_{ESC} is the very well-known escape velocity of an object just to leave the Sun or the planet.

8. Moment of the Tangent Momentum and the Moment of the Normal Momentum of the Newton's Parabola

Based on the formulae in Tables I and in Figures in this contribution we can evaluate the moment of the tangent momentum L_T and to introduce a new physical quantity - the moment of the normal momentum L_N.

The moment of momentum L is defined as the product of the linear momentum with the length of the moment arm, a line dropped perpendicularly from the origin onto the path of the particle. It is this definition: L = (length of moment arm) x (linear momentum).

The moment of the tangent momentum L_T for the Newton's parabola is given as:

$$L_T = m\ v_T JN = m\ v_{esc} \frac{\sqrt{a}}{\sqrt{a+x}} a\ \frac{\sqrt{a+x}}{\sqrt{a}} = m\ v_{esc} a\ = \sqrt{2}\ m\ v_c a \tag{8}$$

where m is the mass of an object, v_T the tangent velocity of an object on the parabolic orbit planet and JN is the length of the moment arm (the distance between the Newton's occupied focus and the tangent to the point F). The moment of the tangent momentum L_T is constant during the complete parabolic path of the Newton's parabola. Therefore, there is no contribution to the torque from this moment of the tangent momentum. This is very well-known experimental fact documented in the existing literature.

The moment of the normal momentum L_N for the Newton's parabola is given as:

$$L_N = m\ v_N NC = m\ v_{esc} \frac{\sqrt{a+x}}{\sqrt{x}} a\ \frac{\sqrt{x}}{\sqrt{a+x}} = m\ v_{esc} a\ = \sqrt{2}\ m\ v_C a \tag{9}$$

where m is the mass of an object, v_N the normal velocity of an object on the parabolic orbit and NC is the length of the moment arm (the distance between the Newton's occupied focus and the tangent to the point C). The moment of the normal momentum is constant during the complete path of the Newton's parabola. Therefore, there is no contribution to the torque from this moment of the normal momentum. This is very well-known experimental fact documented in the existing literature.

9. "Antikythera Mechanism" in the Solar System

We propose to use the very-well known Antikythera Mechanism as an analogy for the visible PAN Parabola - a part of our Aristotelian World - connected deeply with invisible curves from the Plato's Realm - Pappus' Directrix, Apollonius' auxiliary Circle, Newton's Evolute, Castillon's cardioid, Leibniz's six parameters (abscissa, ordinate, length of tangent, subtangent, length of normal, and subnormal), Ptolemy's Circle (Hodograph).

Are there some more hidden curves in the Plato's Realm connected to the PAN Parabola? How to distinguish the real physical meaning written in those curves from fictious events if both are mathematically correct? The detailed analysis of these ideas we will leave to the Readers of this Journal better educated in mathematics.

10. Conclusions

1. We have presented some quantitative properties of the Newton's parabola in Table 1 and Figures 1 - 7.

2. We have studied the interplay of the directrix of parabola **P** (=Pappus) with the vertex of parabola **A** (=Apollonius of Perga) and the occupied focus of parabola **N** (=Newton) in the PAN Parabola.

3. We have discovered a new trigonometric road leading to the Newton's gravitational formula.

4. We have found the expression for the tangent velocity in the Castillon's cardioid.

4. We have employed Leibniz's length of the normal and subnormal to get an expression for the tangent velocity of an object on the parabolic orbit.

5. We have employed Leibniz's length of the tangent and subtangent to get an expression for the normal velocity of an object on the parabolic orbit.

6. We have derived an expression for the moment of tangent momentum of an object on the parabolic orbit.

7. We have derived an expression for the moment of normal momentum of an object on the parabolic orbit.

8. We have used the Ptolemy's Circle as the new Hodograph for parabolic orbits.

9. Are there some more hidden curves in the Plato's Realm connected to the Newton's parabola? How to distinguish the real physical meaning written in those curves from fictious events if both are mathematically correct?

Acknowledgments

This work was supported by the JP&FŠ Agency (Contract Number 25g/1963), by the VZ&MŠ Agency (Contract Number 16000/1989) and by the GMS Agency (Contract Number 69110/1992). JP organized for us the very stimulating seminar on the Feynman's lost lecture (18.12.2018). We were supported by the contract number 28101918/2018. We have found the valuable support on the web site www.wolframalpha.com with the corrections of used formulae.

Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

Abreu, J. L., & Barot, M. (year not specified). *A Geometric Approach to Planetary Motion and Kepler Laws*. Retrieved from http://arquimedes.matem.unam.mx/jlabreu/GeomKepler.pdf

Adams, J. (1818). *The Elements of the Ellipse Together with the Radii of Curvature Relating to that Curve; and of Centripetal and Centrifugal Forces in Elliptical Orbits*. Longman, London. Retrieved from https://books.google.cz/books?id=ckKMj7RIpccC&pg=PA108&lpg=PA108&dq=James+Adams+elements+of+the+ellipse&source=bl&ots=XKw8Iyleis&sig=L2udhh3CosWp9N0Z9lPuFamQDHg&hl=cs&sa=X&ved=0ahUKEwi8oJv57ubbAhWEY1AKHTa8AXoQ6AEIJzAA#v=onepage&q=James%20Adams%20elements%20of%20the%20ellipse&f=false

Aiton, E. (1969). Kepler's Second Law of Planetary Motion. *Isis: A Journal of the History of Science, 60*, 75-90.

Antikythera Mechanism. (2018). Retrieved from https://en.wikipedia.org/wiki/Antikythera_mechanism

Auxiliary Circle of Parabola. (n. d.). Retrieved from https://www3.ul.ie/~rynnet/swconics/ACT%27S.htm

Barbour, J. (2014). Kepler and Mach's Principle. In J. Bičák & T. Ledvinka (Eds.), *General Relativity, Cosmology and Astrophysics. Fundamental Theories of Physics* (Vol. 177). Springer Cham.

Bardini, G., & Gianella, G. M. (2016). A Historical Walk along the Idea of Curvature, from Newton to Gauss Passing from Euler. *International Mathematical Forum, 11*, 259-278. Retrieved from http://www.m-hikari.com/imf/imf-2016/5-8-2016/p/bardiniIMF5-8-2016.pdf

Barker, P. (1994). Distance and Velocity in Kepler's Astronomy. *Annals of Science, 51*, 59-73.

Beckmann, B. (2006). Feymann Says "Newton Implies Kepler, No Calculus Needed". *The Journal of Symbolic Geometry, 1*, 57-72. Retrieved from http://ceadserv1.nku.edu/longa/classes/calculus_resources/docs/kep.pdf

Beech, M. (2005). On Ptolemy's Equant, Kepler's Second Law, and the Non-existent "Empty-Focus" Cometarium. *Journal of the Royal Society of Canada*, 120-123.

Besant, W. H. (2009). *Conic Sections: Treated Geometrically* (9th ed.). Merchant Books. ISBN-10: 1603862579. Retrieved from https://www.gutenberg.org/files/29913/29913-pdf.pdf.

Bičák, J., & Ledvinka, T. (Eds.) (2014). *General Relativity, Cosmology, and Astrophysics*. Perspectives 100 years after Einstein's stay in Prague. Springer. ISBN-10: 3319063480.

Blaschke, P. (2017). *Pedal Coordinates, Dark Kepler and Other Forces Problems*. Arxiv: 1704.00897v1.

Blåsjö, V. (2017). *Transcendental Curves in the Leibnizian Calculus*. Academic Press, Elsevier, ISBN: 978-0-12-813237-1.

Blåsjö, V. (2018). Mathematicians Versus Philosophers in Recent Work on Mathematical Beauty. *Journal of Humanistic Mathematics, 8*, 414-431. Retrieved from https://scholarship.claremont.edu/cgi/viewcontent.cgi?referer=https://www.google.com/&httpsredir=1&article=1380&context=jhm

Böhm, J. (2016). *Wonderful World of Pedal Curves*. Retrieved from http://rfdz.ph-noe.ac.at/fileadmin/Mathematik_Uploads/ACDCA/TIME2016/Boehm_Pedals_.pdf

Brackenridge, J. B. (1989). Newton's Mature Dynamics and the Principia: A Simplified Solution to the Kepler Problem. *Historica Mathematica, 16*, 36-45. Retrieved from https://core.ac.uk/download/pdf/82476251.pdf

Brackenridge, J. B. (1996). *The Key to Newton's Dynamics: The Kepler Problem and the Principia*. Berkeley: University of California Press. ISBN: 978-0520202177.

Bruneau, O. (2015). *ICT and History of Mathematics: The Case of the Pedal Curves from 17th Century to 19th Century*. Retrieved from https://hal.archives-ouvertes.fr/hal-01179909/document

Cardioid: How to cut the Newton's apple? (by Laura Pennington). https://study.com/academy/lesson/cardioid-in-math-definition-equation-examples.html

Cardioid: http://www-history.mcs.st-andrews.ac.uk/Curves/Cardioid.html

Cardioid: http://xahlee.info/SpecialPlaneCurves_dir/Cardioid_dir/cardioid.html

Cardioid: https://en.wikipedia.org/wiki/Cardioid

Cavalieri, B. (1643). *Trigonometria plana, et sphærica, linearis, & logarithmica*. Frontispiece. Retrieved from http://lhldigital.lindahall.org/cdm/ref/collection/emblematic/id/1365

Chandrasekhar, S. (1995). *Newton's Principia for the Common Reader*. Oxford University Press, Oxford, ISBN 978-0-19-852675-9.

Choi, S. C., & Wildberger, N. J. (2018). *The Universal Parabola*. KoG 22, 2018.

Cohen, I. B. (1999). *A Guide to Newton's Principia*. The Principia: The Mathematical Principles of Natural Philosophy. Berkeley, CA, University California Press. ISBN: 978-0-520-08816-0.

Cohen, I. B., & Smith, G. E. (Eds.) (2004). *The Cambridge Companion to NEWTON*. Cambridge University Press, Cambridge, ISBN 0-521-65696-6.

Colwell, P. (1993). *Solving Kepler's Equation Over Three Centuries*. Willmann-Bell. Inc. Richmond. ISBN 0-943396-40-9.

Darrigol, O. (2012). *A History of Optics*. From Greek Antiquity to the Nineteenth Century. Oxford University Press. Oxford. ISBN-10: 0199644373.

Dennis, D. (1995). *Historical Perspectives for the Reform of Mathematics Curriculum: Geometric Curve Drawing Devices and their Role in the Transition to an Algebraic Description of Functions*. Dissertation at Cornell University. Retrieved from http://www.quadrivium.info/mathhistory/CurveDrawingDevices.pdf

Dennis, D., & Confrey, J. (1995). Functions of a Curve: Leibniz's Original Notion of Functions and its Meaning for the Parabola. *The College of Mathematics Journal*, 124-130. Retrieved from http://www.quadrivium.info/mathhistory/Parabola.pdf

Derbes, D. (2001). Reinventing the wheel: Hodographic solutions to the Kepler problems. *Am. J. Phys., 69*, 481-489.

Donahue, W. (1994). Kepler's Invention of the Second Planetary Law. *British Journal fort he History of Science, 27*, 89-102.

Einstein, A. (1918). Prinzipielles zur Allgemeinen Relativitätstheorie. *Annalen der Physik, 360*, 241.

Feynman, R. F. (1963). *The Character of Physical Law, Part 1, The Law of Gravitation ("How should angels generate the Kepler ellipse?")*. Retrieved from https://www.youtube.com/watch?v=j3mhkYbznBk

Fried, M., & Unguru, S. (2001). *Apollonius of Perga's Conica: Text, Context, Subtext*. Mnemosyne, Bibliotheca Classic. ISBN-10: 9004119779.

Gant, de F. (1995). *Force and Geometry in Newton's Principia*. Princeton University Press, Princeton. ISBN 0-691-03367-6.

Gingerich, O. (1973). From Copernicus to Kepler: Heliocentrism as Model and as Reality. *Proceedings of the American Philosophical Society, 117*, 513-522.

Glaser, G., Stachel, H., & Odehnal, B. (2016). *The Universe of Conics: From Ancient Greeks to 21st Century Developments*. Springer Spektrum. ISBN-10: 36622454491.

Goodstein, D., & Goodstein, J. R. (2000). *Feynman's Lost Lecture*. W.W. Norton & Company. ISBN-10: 0393319954.

Guicciardini, N. (2003). *Reading the Principia: The Debate on Newton's Mathematical Methods for Natural Philosophy from 1687 to 1736*. Cambridge University Press, Cambridge. ISBN-10: 0521544033.

Hadot, P. (2008). *The Veil of Isis: An Essay on the History of the Idea of Nature*. Belknap Press. ISBN-10: 0674030494.

Hall, A. R. (2015). *Philosophers at War: The Quarrel Between Newton and Leibniz*. Cambridge University Press. ISBN: 052152489X.

Hamilton, W. R. (1847). The Hodograph, or a New Method of Expressing in Symbolical Language the Newtonian Law of Attraction. *Proceedings of the Royal Irish Academy, 3*, 344-353. Retrieved from https://www.emis.de/classics/Hamilton/Hodo.pdf

Hatfield, B. (Ed.) (2018). *Feynman Lectures on Gravity*. CRC Press. ISBN 13: 978-0-8133-4038-8 (pbk).

Hawking, S. W., & Israel, W. (Eds.) (1989). *Three Hundred Years of Gravitation*. Cambridge University Press, Cambridge. ISBN-10: 0521379768.

Heath, T. L. (2015). *Apollonius of Perga: Treatise on Conic Sections*. Carruthers Press. ISBN-10: 1446021262.

Horský, Z. (1980). *Kepler in Prague*. Mír. Prague. 601/22/85.5 (In Czech).

Hsiang, W. Y., & Straume, E. (2014). *Revisiting the mathematical synthesis of the laws of Kepler and Galileo leading to Newton's law of universal gravitation*. Arxiv: 1408.6758v1.

Huygens, Ch. (1659). *On Centrifugal Force*. (De Vi Centrifuga). Translated by M.S. Mahonay. Retrieved from https://www.princeton.edu/~hos/mike/texts/huygens/centriforce/huyforce.htm

Kardioide (Herzkurve). Retrieved from https://de.wikipedia.org/wiki/Kardioide

Kawarabuki, H., & Nakazato, H. (2013). Another Question on Kepler's so Called Distance Law. *Int. Math. Forum, 8*, 59-63. Retrieved from https://pdfs.semanticscholar.org/3d20/aa16678717a8ceb1062056a37de00365faa7.pdf

Kepler, J. (1609). *Astronomia Nova*. Translated by Max Casper. Marix Verlag. ISBN-10:3865390145.

Kepler, J. (1619). *Harmony of Worlds*. Translated by Ch. G. Wallis, Ecyclopædica Britannica, Chicago.

Kepler, J. (1627). *Tabulaœ Rudolphinaœ*. Retrieved from https://bibdig.museogalileo.it/Teca/Viewer?an=334726

Köller, J. (2016). *Fusspunktkurven und Gegenfusspunktkurven*. Retrieved from http://www.mathematische-basteleien.de/fusspunktkurve.htm

Laird, W. R., & Roux, S. (Eds.) (2008). *Mechanics and Natural Philosophy before the Scientific Revolution*. Springer. ISBN 978-90-481-7491-1.

Lockwood, E. H. (1961). *A Book of Curves*. Cambridge University Press. Retrieved from http://www.aproged.pt/biblioteca/ABookofCurvesLockwood.pdf

Maor, E., & Jost, E. (2014). *Beautiful Geometry*. Princeton University Press, Princeton. ISBN-13:978-0-691-15099-4.

Markowsky, G. (2011). A Retelling of Newton's Work on Kepler's Laws. *Expositiones Mathematicae, 29*, 253-282. Retrieved from https://www.sciencedirect.com/science/article/pii/S0723086911000089

Menaechmus: https://en.wikipedia.org/wiki/Menaechmus

Miller, D. M. (2008). "O Male Factum": Rectilinearity and Kepler's Discovery of the Ellipse. *Journal for the History of Astronomy, 39*, 43-63. Retrieved from http://adsbit.harvard.edu/cgi-bin/nph-iarticle_query?2008JHA....39...43M&defaultprint=YES&filetype=.pdf

Nauenberg, M. (1993). *Newton's Early Computational Method for Dynamics*. Retrieved from https://core.ac.uk/download/pdf/82476251.pdf

Nauenberg, M. (1994). Hooke, Orbital Motion, and Newton's Principia. *Am. J. Phys., 62*, 331.

Nauenberg, M. (2018). *A Simple Derivation of the Two Force Laws for Elliptic Orbits from Proposition 6 in Newton's Principia*. Retrieved from https://arxiv.org/pdf/1805.08872.pdf

Nauenberg, M. (2018). Newton's Graphical Method for Central Force Orbits. *Am. J. Phys., 86*, 765-771.

Nauenberg, M. (2018). *Visiting Newton's Atelier before the Principia*, 1679-1684. Arxiv: 1805.06871v.

Newton, I. (1687). The Principia. *Mathematical Principles of Natural Philosophy*. Translated by I.B. Cohen and A. Whitman. University California Press, Berkeley. ISBN 978-0-520-08816-0.

Newton's Law of Universal Gravitation. Retrieved from https://en.wikipedia.org/wiki/Newton%27s_law_of_universal_gravitation

Papaspirou, P., & Moussas, X. (2013). *The Hellenistic and Alexandrian Influences on Johannes' Kepler Work*. Retrieved from https://pdfs.semanticscholar.org/fc7f/ab1980955f87e2a53622ceaa0c1ee75e4e60.pdf?_ga=2.199516460.368964894.1539953979-1077201457.1539953979

Pappus of Alexandria: https://en.wikipedia.org/wiki/Pappus_of_Alexandria

Parabel Eigenschaften: https://de.wikipedia.org/wiki/Parabel_(Mathematik)

Parabola Properties: http://mathworld.wolfram.com/Parabola.html

Parabola Properties: http://www.nabla.hr/IA-ParabolaAndLine2.htm

Parabola Properties: https://en.wikipedia.org/wiki/Parabola

Pedal curves properties. (2018): https://en.wikipedia.org/wiki/Pedal_curve

Pennington, L. (2019). *Cardioid in Math: Definition, Equation & Examples*. How to cut the Newton's Apple? Retrieved from https://study.com/academy/lesson/cardioid-in-math-definition-equation-examples.html

Podolský, J. (2018). *Feynman Lost Lecture (In Czech)*. Retrieved from https://www.youtube.com/watch?v=Un0MOt7o6R8

Ptolemy's Almagest (translated by G. J. Toomer & F. O. Gingerich) (1998). Princeton University Press, Princeton. ISBN-10: 0-691-00260-6.

Roveli, C. (2018). Physics Needs Philosophy. Philosophy Needs Physics. *Found. Phys., 48*, 481-491. Retrieved from https://arxiv.org/ftp/arxiv/papers/1805/1805.10602.pdf

Russell, J. L. (1964). Kepler's Laws of Planetary Motion: 1609 - 1666. *The British Journal for the History of Science, 2*, 1 -24.

Rynne, S. (2006). *Curvature of the Ellipse*. Retrieved from https://www3.ul.ie/~rynnet/swconics/E-COC.htm

Schmarge, K. (1999). *Conic Sections in Ancient Greece*. Retrieved from http://sites.math.rutgers.edu/~cherlin/History/Papers1999/schmarge.html

Sonar, T. (2018). *The History of the Priority Diſpute between Newton and Leibniz*. Birkhäuser, ISBN: 9783319725611.

Stávek, J. (2018). Kepler's Ellipse Generated by the Trigonometrically Organized Gravitons. *Appl. Phys. Res., 10*, 26-37.

Stávek, J. (2018). Kepler's Ellipse Observed from Newton's Evolute (1687), Horrebow's Circle (1717), Hamilton's Pedal Curve (1847), and Two Contrapedal Curves (28.10.2018). *Appl. Phys. Res., 10*, 90-101.

Stávek, J. (2019). Newton's Hyperbola Observed from Newton's Evolute (1687), Gudermann's Circle (1833), the Auxiliary Circle (Pedal Curve and Inversion Curve), the Lemniscate of Bernoulli (1694) (Pedal Curve and Inversion Curve) (09.01.2019). *Appl. Phys. Res., 11*, 65-78.

Strutz, Ch. (2001). *Über das Latus Rectum Principalis in Newton's Himmelmechanik*. Retrieved from http://www.schulphysik.de/strutz/latrect1.pdf

Strutz, Ch. (2001). *Von Apollonius zur Himmelsmechanik*. Retrieved from http://www.schulphysik.de/strutz/gravit2.pdf

Subtangent and Subnormal: http://www.expertsmind.com/learning/subtangent-and-subnormal-assignment-help-7342874322.aspx

Subtangent and Subnormal: https://www3.ul.ie/~rynnet/swconics/PP%27S.htm

Subtangent: https://en.wikipedia.org/wiki/Subtangent

Sugimoto, T. (2009). How to Present the Heart of Newton's Principia to the Layperson: A Primer on the Conic Sections without Apollonius of Perga. *Symmetry: Culture and Science, 20*, 113-144. Retrieved from http://www.is.kanagawa-u.ac.jp/overview/doc/Symmetry_Sugimoto_2009.pdf

Suzuki, M. S., & Suzuki, I. S. (2015). *Hodographic Solutions to the Kepler's problem*. Retrieved from https://www.researchgate.net/profile/Masatsugu_Suzuki/publication/271763295_Hodographic_Solutions_to _the_Kepler%27s_Problem_Masatsugu_Sei_Suzuki_and_Itsuko_S_Suzuki/links/54d0d2720cf298d656691a67/Hodographic-Solutions-to-the-Keplers-Problem-Masatsugu-Sei-Suzuki-and-Itsuko-S-Suzuki.pdf

Švejda, A. (2004). Kepler and Prague. *National Technical Museum*. ISBN: 80-7037-130-7.

Swetz, F. (2016). *Mathematical Treasure: Halley's Conics of Apollonius*. Convergence. Retrieved from https://www.maa.org/press/periodicals/convergence/mathematical-treasure-halleys-conics-of-apollonius

Tan, A. (2008). *Theory of Orbital Motion*. World Scientific Publishing Co., Singapore. ISBN-10: 981-270-911-8.

Todhunter, I. (1881). *A Treatise on Plane Co-ordinate Geometry as Applied to the Straight Line and the Conic Sections*. MacMillan and Co. London. Retrieved from https://projecteuclid.org/euclid.chmm/1263315392

Wolfram, S. (2018). *The History and Future of Special Functions*. Retrieved from https://www.stephenwolfram.com/publications/history-future-special-functions/

Yates, R. C. (1974). *Curves and their Properties*. National Council of Teachers of Mathematics, ISBN: 10-087353039X.

Zeuthen, H. G. (1896). *Die Lehre von den Kegelschnitten im Altertum*. Kopenhagen, A.F. Höst & Son. Retrieved from https://ia800208.us.archive.org/20/items/dielehrevondenk00zeutgoog/dielehrevondenk00zeutgoog.pdf