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Non-standard Numerical Approaches for Solutions of Some Second Order Ordinary Differential Equations

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Short Research Article

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Abstract

A set of non-standard discrete models are constructed for the solution of non-homogenous second order equation. The method of non-local approximation and renormalization of the discretization functions have been applied to some examples and the result has shown that the schemes behave qualitatively like the original equation.

Keywords: Nonstandard methods; renormalized denominator functions; initial value problems; non-local approximation; discrete models.

1 Introduction

Some studies of traditional standard numerical methods have shown that numerical instabilities exist to the use of these methods [1].

Non-standard method therefore came as a result of the instabilities noted in other earlier standard methods, for example: Selection of grid size can be a problem because schemes created using standard methods exhibit different behavioral patterns for different sizes of h. In fact, it has been shown that Euler central scheme are unstable. For the decay equation the central difference scheme has numerical instabilities for all step-size values; the forward Euler schemes provide useful discrete models if limitations are placed on the step-size; and the backward Euler scheme can be used for any (positive) step-size. Except for the central difference scheme, the other three

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discrete models will give excellent quantitative numerical solutions if h is made small enough, i.e., $0 < h \ll 1$ [2].

It has been shown by [3,4] that the central difference scheme allows for the existence of chaotic orbits for all positive time-steps for the Logistic differential equation. Notable work on this problem has been done by other researchers including [5,6,7,8]. The major conclusion is that the use of the central difference scheme forces all the fixed-points to become unstable.

The use of the standard rules does not lead to a unique discrete model. Consequently, one of the questions before us is which, if any, of the standard finite-difference schemes should be used to obtain numerical solutions for a particular differential equation? Another very important issue is the relationship between the solutions to a given discrete model and that of the corresponding differential equation.

In the paper we will construct qualitatively stable models of some non-homogeneous second order ordinary differential equation by applying the set of rules suggested by [1,7].

Consider an ordinary differential equation of the form

$$y'' + p(x)\, y' + Q(x)\, y = f(x) \quad (1)$$

where p(x), Q(x) and f(x) are non-zero real valued function of (x).

We will apply rule 2 & 3 (see [1]) to each of the components of the equations as shown below

$$y'' \equiv \frac{y_{k+1}-2y_{k+}\, y_{k-1}}{\varphi} \quad \text{where} \quad \varphi(h) \quad \to h^2 + 0(h^4)\ as\ h \ \to 0 \quad (2)$$

$$y' \equiv \frac{(y_{k+1}-y_k)}{\psi} \quad \text{where} \quad \psi(h) \quad \to h + 0(h^2)\ as\ h \ \to 0 \quad (3)$$

$$y' \equiv \frac{(y_{k+1}-\beta y_k)}{\psi} \quad \text{where} \quad \psi(h) \quad \to h + 0(h^2),\ \beta(h) \to 1\ as\ h \ \to 0 \quad (4)$$

2 Analysis of the Non-standard Discrete Models

The general second order ordinary differential equation (1) can be transformed as shown below:

$$y'' + p(x)\, y' + Q(x)\, y = f(x)$$
$$\frac{y_{k+1}-2y_{k+}\, y_{k-1}}{\varphi} + p(x_k)\, \frac{(y_{k+1}-y_k)}{\psi} + Q(x_k)\, y_k = f(x_k) \quad (5)$$

or

$$\frac{y_{k+1}-2y_{k+}\, y_{k-1}}{\varphi} + p(x_k)\, \frac{(y_{k+1}-\beta y_k)}{\psi} + Q(x_k)\, y_k = f(x_k) \quad (6)$$

3 Example I (see [9])

$$y'' - \frac{2}{x}\, y' + \frac{(x^2+2)}{x^2} y - xe^x = 0 \quad (7)$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{4}\left(8 + e^{\frac{\pi}{2}}\right),\ \dot{y}\left(\frac{\pi}{2}\right) = 4 - \pi + \frac{1}{4}(\pi + 2)e^{\frac{\pi}{2}} y(x) = 2xcosx + 4xsinx + \frac{1}{2}xe^x \quad (6)$$

3.1 Scheme (A_1)

Applying the transformation equations (2) and (3) in (7) we have the following

$$\frac{y_{k+1}-2y_k+y_{k-1}}{\varphi}=\frac{2}{x}\,\frac{(y_{k+1}-y_k)}{\psi}-\frac{(x^2+2)}{x^2}y_k+\,xe^x \tag{7}$$

$$y_{k+1}\frac{(\psi x-2\varphi)}{\psi x}=\left(2-\frac{2}{\psi x}-1-\frac{2}{x^2}\right)\varphi y_k+\,x\varphi e^x-\varphi y_{k-1} \tag{8}$$

$$y_{k+1}=A\left(\frac{(2\psi x^2-2x-\psi x^2-2\psi)}{\psi x^2}\right)\varphi y_k+A\,x\varphi e^x-A\varphi y_{k-1} \tag{9}$$

Where $A=\frac{\psi x}{\psi x-2\varphi}$, $x=kh,\ x^2=(kh)^2+kh^2+\frac{h^2}{4}$

using $=4sin^2(^h/_2)$,ψ =sin (h) ,

3.2 Scheme (A_2)

Applying the transformation equations (2) and (4) in (7) we have the following

$$\frac{y_{k+1}-2y_k+y_{k-1}}{\varphi}=\frac{2}{x}\,\frac{(y_{k+1}-\beta y_k)}{\psi}-\frac{(x^2+2)}{x^2}y_k+\,xe^x \tag{10}$$

$$y_{k+1}\frac{(\psi x-2\varphi)}{\psi x}=\left(2-\frac{2\beta}{\psi x}-1-\frac{2}{x^2}\right)\varphi y_k+\,x\varphi e^x-\varphi y_{k-1} \tag{11}$$

$$y_{k+1}=A\left(\frac{(2\psi x^2-2\beta x-\psi x^2-2\psi)}{\psi x^2}\right)\varphi y_k+A\,x\varphi e^x-A\varphi y_{k-1} \tag{12}$$

Where $A=\frac{\psi x}{\psi x-2\varphi}$, $x=kh,\ x^2=(kh)^2+kh^2+\frac{h^2}{4}$

And $\varphi=4sin^2(^h/_2)$,ψ =sin (h) , $\beta=cos(h)$

3.3 Scheme (A_3)

Modify the scheme (A2) by changing the denominator function ψ in equation (12)

$$y_{k+1}=A\left(\frac{(2\psi x^2-2\beta x-\psi x^2-2\psi)}{\psi x^2}\right)\varphi y_k+A\,x\varphi e^x-A\varphi y_{k-1} \tag{13}$$

$$A=\frac{\psi x}{\psi x-2\varphi},\ x=kh,\ x^2=(kh)^2+kh^2+\frac{h^2}{4} \tag{14}$$

Using $\varphi=4sin^2(^h/_2)$,$\psi=\frac{(e^{\lambda h}-1)}{\lambda}$,$\lambda \in R$, $\beta=cos(h)$

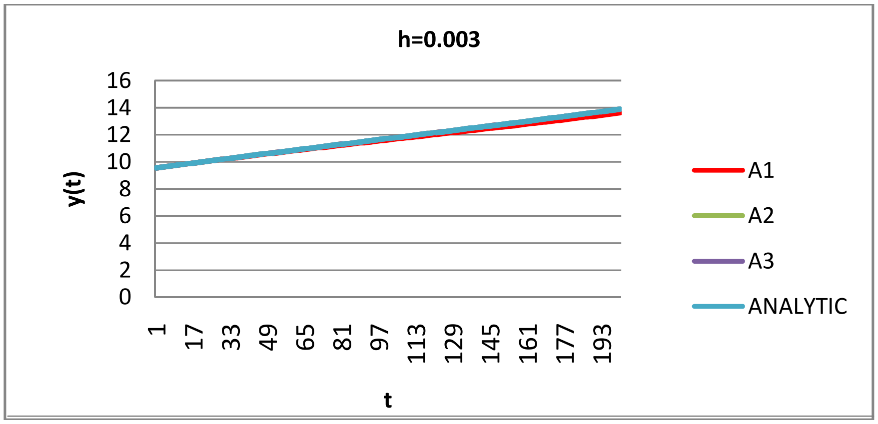

Fig. 1. Graph of the schemes of $y'' - \frac{2}{x}y' + \frac{(x^2+2)}{x^2}y - xe^x = 0$ **for h=0.003**

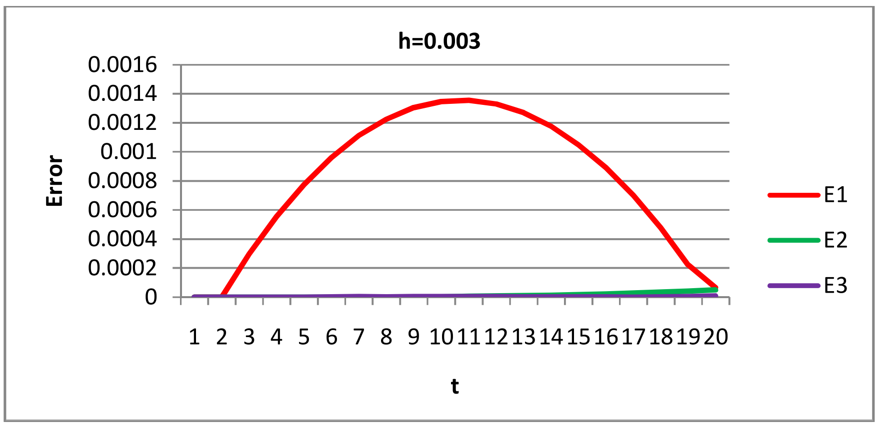

Fig. 2. Graph of the errors of deviation of the schemes of $y'' - \frac{2}{x}y' + \frac{(x^2+2)}{x^2}y - xe^x = 0$

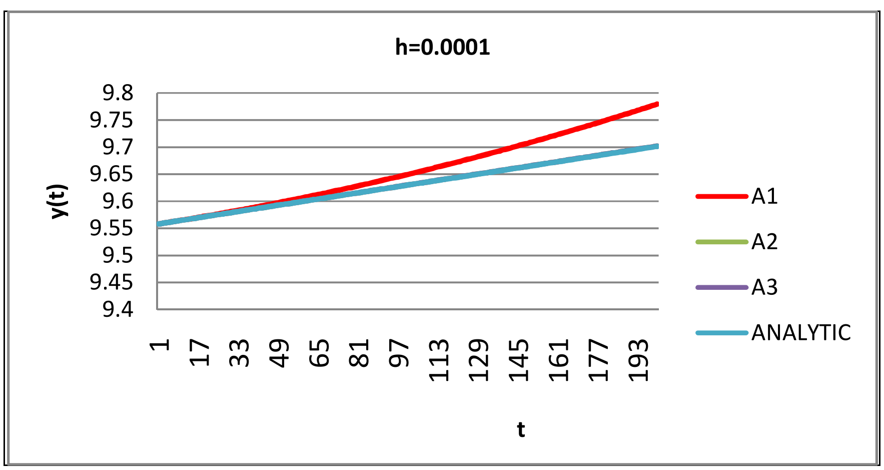

Fig. 3. Graph of the schemes of $y'' - \frac{2}{x}y' + \frac{(x^2+2)}{x^2}y - xe^x = 0$ **for h=0.0001**

Table 1. Result and error of schemes A_1, A_2, A_3 for h=0.003

T	A1	A2	A3	Analytic	E1	E2	E3
1.5	9.558448	9.558448	9.558448	9.558448	0	0	0
1.503	9.580832	9.580832	9.580832	9.580832	0	0	0
1.506	9.603494	9.603198	9.603198	9.603198	0.000295639	0	0
1.509	9.626101	9.625547	9.625547	9.625546	0.000554085	9.53675E-07	9.53675E-07
1.513	9.648654	9.647879	9.64788	9.647879	0.000775337	0	9.53674E-07
1.516	9.671154	9.670193	9.670195	9.670193	0.000961304	0	1.90735E-06
1.519	9.693601	9.69249	9.692492	9.692489	0.001111984	9.53674E-07	3.8147E-06
1.522	9.715994	9.714768	9.714772	9.714769	0.001224518	9.53674E-07	2.86102E-06
1.525	9.738335	9.73703	9.737035	9.737031	0.001303673	9.53674E-07	3.8147E-06
1.528	9.760623	9.759274	9.75928	9.759276	0.001346588	2.86102E-06	3.8147E-06
1.531	9.782859	9.781499	9.781508	9.781505	0.001354218	5.72205E-06	3.8147E-06
1.534	9.805043	9.803707	9.803719	9.803715	0.001328468	7.62939E-06	3.8147E-06
1.538	9.827177	9.825898	9.825912	9.825909	0.001268387	1.04904E-05	2.86102E-06
1.556	9.958935	9.958672	9.958709	9.958713	0.000222206	4.1008E-05	3.8147E-06
1.559	9.980722	9.980739	9.980781	9.980788	6.58035E-05	4.95911E-05	7.6294E-06

Note: E1 is the column for error of deviation of A1 from Analytic solution, E2 is the column for error of deviation of A2 from Analytic solution and E3 is the column for error of deviation of A3 from Analytic solution

Table 2. Result and error of schemes A_1, A_2, A_3 for h=0.0001

t	A1	A2	A3	Analytic	E1	E2	E3
1.5	9.558448	9.558448	9.558448	9.558448	0	0	0
1.5001	9.559165	9.559165	9.559165	9.559165	0	0	0
1.5002	9.559896	9.559882	9.559882	9.559881	1.43051E-05	9.53674E-07	9.53674E-07
1.5003	9.56063	9.560599	9.560599	9.560598	3.14713E-05	9.53674E-07	9.53674E-07
1.5004	9.561368	9.561316	9.561316	9.561314	5.43594E-05	2.86102E-06	2.86102E-06
1.5005	9.56211	9.562034	9.562034	9.562031	7.9155E-05	2.86102E-06	2.86102E-06
1.5006	9.562856	9.562751	9.562751	9.562747	0.000108719	3.8147E-06	3.8147E-06
1.5007	9.563605	9.563468	9.563468	9.563464	0.000141144	3.8147E-06	3.8147E-06
z							
1.5008	9.564359	9.564185	9.564185	9.56418	0.000178337	4.76837E-06	4.76837E-06

1.5009	9.565116	9.564902	9.564902	9.564897	0.000219345	5.72205E-06	5.72205E-06
1.501	9.565877	9.565619	9.565619	9.565613	0.000264168	6.67572E-06	6.67572E-06
1.5011	9.566642	9.566337	9.566337	9.566329	0.000312805	7.6294E-06	7.6294E-06
1.5012	9.56741	9.567054	9.567054	9.567045	0.000365257	8.58307E-06	8.58307E-06
1.5013	9.568183	9.567771	9.567771	9.567761	0.000421524	9.53674E-06	9.53674E-06
1.5014	9.568959	9.568488	9.568488	9.568479	0.000480652	9.53674E-06	9.53674E-06
1.5015	9.569739	9.569205	9.569205	9.569195	0.000544548	1.04904E-05	1.04904E-05
1.5016	9.570523	9.569922	9.569922	9.569911	0.000612259	1.14441E-05	1.14441E-05
1.5017	9.571311	9.57064	9.57064	9.570627	0.000683784	1.23978E-05	1.23978E-05
1.5018	9.572103	9.571357	9.571357	9.571343	0.000759125	1.33514E-05	1.33514E-05
1.5019	9.572898	9.572074	9.572074	9.572059	0.000839233	1.52588E-05	1.52588E-05

Table 3. Result and error of schemes B_1, B_2, B_3 for h=0.1/40

T	B1	B2	B3	Analytic	E1	E2	E3
1	2	2	2	2	0	0	0
1.0025	2.025078773	2.025078773	2.025078773	2.025078773	0	0	0
1.005	2.050312996	2.050313711	2.050312996	2.050312996	0	7.15255E-07	0
1.0075	2.075703144	2.07570529	2.075703144	2.075705767	2.6226E-06	4.76837E-07	2.6226E-06
1.01	2.101249695	2.101253748	2.101249695	2.101254702	5.0068E-06	9.53674E-07	5.00679E-06
1.0125	2.126953363	2.126959801	2.126952887	2.126963139	9.7752E-06	3.33786E-06	1.0252E-05
1.015	2.152814388	2.152823687	2.152813196	2.152828693	1.4305E-05	5.00679E-06	1.54972E-05
1.0175	2.178833246	2.178845882	2.1788311	2.178854465	2.1219E-05	8.58307E-06	2.3365E-05
1.02	2.205010653	2.205026865	2.205006838	2.205038548	2.7895E-05	1.16825E-05	3.17097E-05
1.0225	2.231346846	2.231367111	2.231341124	2.231383562	3.6716E-05	1.64509E-05	4.24385E-05
1.025	2.257842541	2.257867098	2.257834435	2.25788784	4.53E-05	2.07424E-05	5.34058E-05
1.0275	2.284498453	2.284527302	2.284487009	2.284554243	5.579E-05	2.69413E-05	6.7234E-05
1.03	2.31131506	2.3113482	2.311299324	2.311380386	6.5327E-05	3.21865E-05	8.10623E-05

Note: E1 is the column for error of deviation of B1 from Analytic solution, E2 is the column for error of deviation of B2 from Analytic solution and E3 is the column for error of deviation of B3 from Analytic solution

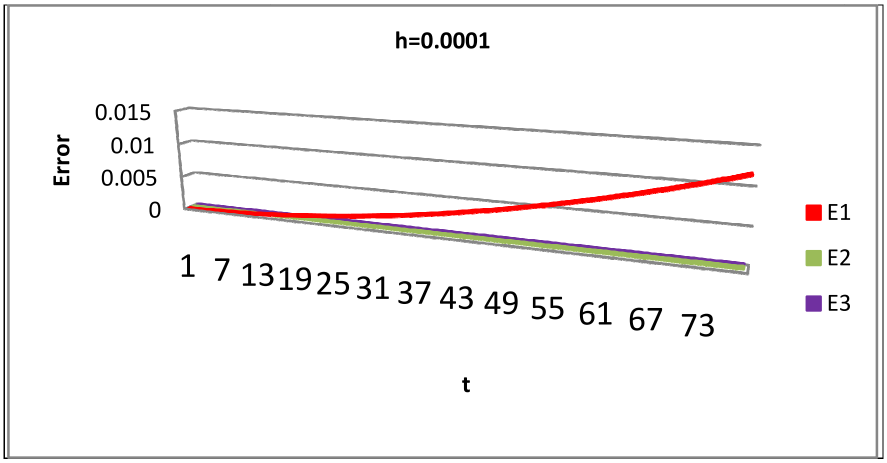

Fig. 4. Graph of the errors of deviation of the schemes for h=0.0001

4 Example II (see [9])

$$y'' - \frac{3}{x}y' + 3y - 2x = 0 \tag{15}$$

$$y(0) = 2, y'(0) = 10 \qquad y(x) = 3x^3 - 2x + x^2 + x^2 logx \tag{16}$$

4.1 Scheme (B_1)

Applying the transformation equations (2),(3) in (15) we have the following

$$\frac{y_{k+1}-2y_k+y_{k-1}}{\varphi} = 2x + \frac{3}{x}\frac{(y_{k+1}-y_k)}{\psi} - 3(by_k + ay_{k+1}) \tag{17}$$

$$y_{k+1} = 2y_k - y_{k-1} + \frac{3\varphi}{\psi x}y_{k+1} - \frac{3\varphi y_k}{\psi x} - 3\varphi b y_k - 3a\varphi y_{k+1} + 2\varphi x \tag{18}$$

$$y_{k+1}\left(1 - \frac{3\varphi}{\psi x} - 3a\varphi\right) = \left(2 - 3\varphi b - (\frac{3\varphi}{\psi x})\right)y_k - y_{k-1} + (2\varphi x) \tag{19}$$

$$y_{k+1} = \frac{\left(2-3\varphi b-(\frac{3\varphi}{\psi x})\right)y_k - y_{k-1} + (2\varphi x)}{\left(1-\frac{3\varphi}{\psi x} - 3a\varphi\right)} \tag{20}$$

Using $\varphi = 4sin^2({}^h/_2)$,ψ =sin (h)

4.2 Scheme (B_2)

Applying the transformation equations (2),(4) in (15) we have the following

$$\frac{y_{k+1}-2y_k+y_{k-1}}{\varphi} = 2x + \frac{3}{x}\frac{(y_{k+1}-\beta y_k)}{\psi} - 3(by_k + ay_{k+1}) \tag{21}$$

$$y_{k+1} = 2y_k - y_{k-1} + \frac{3\varphi}{\psi x}y_{k+1} - \frac{3\varphi\beta y_k}{\psi x} - 3\varphi b y_k - 3a\varphi y_{k+1} + 2\varphi x \tag{22}$$

$$y_{k+1}\left(1 - \frac{3\varphi}{\psi x} - 3a\varphi\right) = \left(2 - 3b\varphi - (\frac{3\varphi\beta}{\psi x})\right)y_k - y_{k-1} + (2\varphi x) \tag{23}$$

$$y_{k+1} = \frac{\left(2-3b\varphi-\left(\frac{3\varphi\beta}{\psi x}\right)\right)y_k - y_{k-1} + (2\varphi x)}{\left(1-\frac{3\varphi}{\psi x}-3a\varphi\right)} \qquad (24)$$

Using $x = kh,\ x^2 = (kh)^2 + kh^2 + \frac{h^2}{4}$

And $\varphi = 4sin^2(^h/_2)$,ψ =sin (h) , $\beta = cos(h)$

4.3 Scheme (B_3)

Modify the scheme (B_2) by changing the denominator function ψ in equation (24)

$$y_{k+1} = \frac{\left(2-3b\varphi-\left(\frac{3\varphi\beta}{\psi x}\right)\right)y_k - y_{k-1} + (2\varphi x)}{\left(1-\frac{3\varphi}{\psi x}-3a\varphi\right)} \qquad (25)$$

Using $x = kh,\ x^2 = (kh)^2 + kh^2 + \frac{h^2}{4}$

And $\varphi = 4sin^2(^h/_2)$,$\psi = \frac{(e^{\lambda h}-1)}{\lambda}$,$\lambda \in$ R , $\beta = cos(h)$

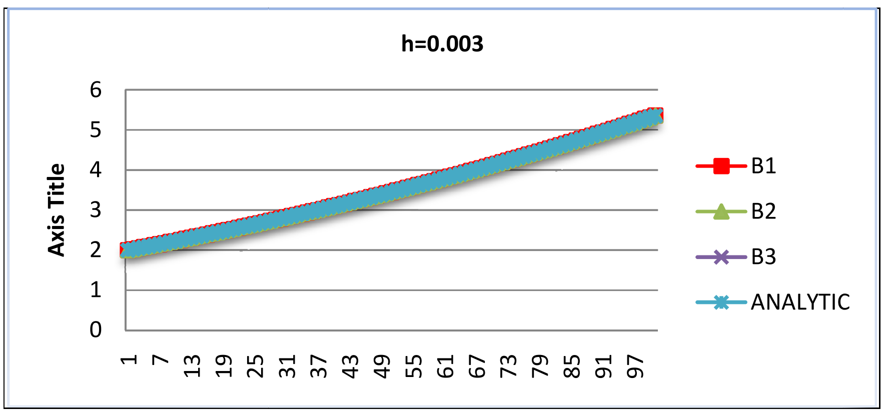

Fig. 5. Graph of the schemes of $y'' - \frac{3}{x}y' + 3y - 2x = 0$ for h=0.0025

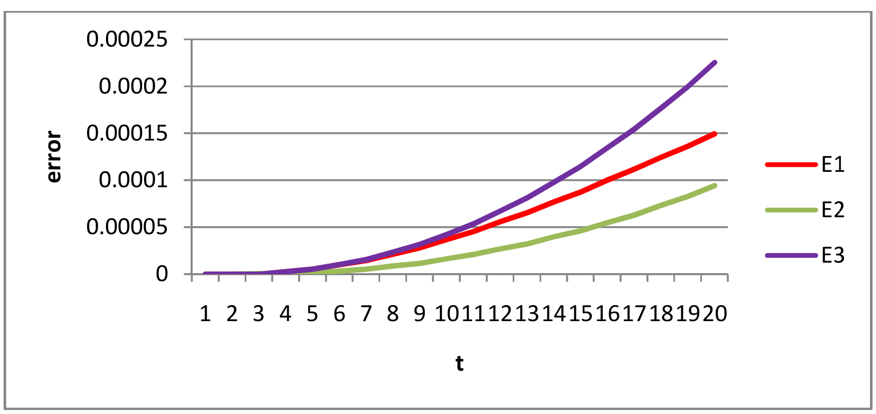

Fig. 6. Graph of the errors of deviation of the schemes for h=0.0025

5 Observations and Conclusion

It can be observed that all the schemes derived display the same dynamics as the analytic solution (see Figs. 1, 3 and 5). The schemes are also stable with respect to monotonicity of solution. The renormalization of the discretization functions through the introduction of β has led to significant improvement of the schemes because the error of approximation reduced considerably (see curves of schemes A_2,A_3, and their corresponding errors presented in Figs. 2 and 4 and Tables 1 and 2. The absolute errors generated for scheme A1 are larger). (see also curves of B_2,B_3 and their corresponding errors presented in Fig. 6 and Table 3. The absolute errors generated for scheme B1 are larger). The choice of normalized denominator can make some difference. It has been observed in the tested cases that the use of $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\lambda \in \mathrm{R}$ is better than $\psi = \sin(h)$ (see Table 1 for Schemes A2 and A3, Table 2 for Schemes A2 and A3, Table 3 for Schemes B2 and B3).This is not unconnected with the opportunity to choose λ appropriately to satisfy the condition $\psi(h) \to h + 0(h^2)\ as\ h \to 0$. This is also because the step size is dynamically varied during the iterations .This confirms some earlier results (for example see [7] and [8]). It can be shown that using a fixed h during iterations makes each of the schemes perform poorly. We can conclude here that the renormalization of the discrete derivatives and the use of normalized and dynamic denominator functions create schemes that have the same dynamics as the original equation .However the complex activity of choosing the best combination of parameters and functions still depend on the experience of the modeler and a fair knowledge of the dynamics of the modeled differential equation. There are no definite rules yet on how to select this variables and functions. The concept of best schemes may be achieved if this riddle is resolved.

Competing Interests

Author has declared that no competing interests exist.

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