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The Conceprts of Delay Differential Equations and IT'S Application

Adamu Wakili[1*]

[1]*Department of Mathematical Sciences, Adamawa State University, Mubi, Adamawa State, Nigeria.*

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Abstract

All processes take time to complete. While physical processes such as acceleration and deceleration take little time compared to the times need to travel most distances, the times involved in biological processes such as gestation and maturation can be substantial when compared to the data-collection times in most population studies. Therefore, it is often imperative to explicitly incorporate these process times in mathematical models of population dynamics. These process times are often called delay times, and the model that incorporate such delay times are referred as delay differential equation (DDE) models.
The models will examine some theoretical concepts and their applications to real life situation. The application examines measles and the time it takes to manifest or to its removal or treatment from the system. The solutions of the models will be displayed in graphical forms using MATLAB method. The analysis of the models indicate the times delay and its characteristics.

Keywords: Time delay, maturation, gestation, oscillation, periodic, parameter and dynamics.

1 Introduction

The use of ordinary and partial differential equations to model biological systems has a long history. These systems cannot capture the rich variety of dynamic behaviour observed in natural systems. This usually lead to systems with more differential equations and having many parameters which cannot be determine experimentally. With the introduction of time delay terms in the differential equations, this area is gaining ground more rapidly than expected. The delay or lags can represent gestation times, incubation periods, transport delays or can simply lead to complicated biological processes together, accounting only for the time required for process to occur.

These models have the advantage of combining a simple, intuitive derivation with a wide variety of possible behaviour rigid for a single system. On the otherwise, these models habour much of the detailed working of complex biological systems and sometimes these details are of interest.

**Corresponding author: adamou_wakili@yahoo.com;*

Recently delay models are becoming more popular, appearing in many branches of biological modeling. They have been used for describing several aspects of infectious disease dynamics; primary infection [1], drug therapy [2] and immune response [3]. Delays have also been extended to the study of chemostat models [4], circadian rhythms [5], epidemiology [6], the respiratory systems [7], tumor growth [8] and neural networks [9].

In many species of population dynamics delay models have been applied in Statistical analysis of ecology data [10, 11].

1.1 Basic Properties of Delay Differential Equations

Like ordinary differential equations, delay differential equations have several features which make their analysis more complicated.

Consider the following delay differential equation

$$\dot{x}(t) = f(x(t), x(t-\tau) \qquad \text{… (1.1)}$$

To begin with an initial value problem requires more information than an analogous problem for a system without delay. For an ordinary differential system, a unique solution is determined by an initial point in Euclidean space at an initial time t_0. For a delay differential system, one requires information on the entire interval $[t_0 - \tau, t_0]$.

To know the rate of change at t_0, one needs to know $x(t_0)\, and\, x(t_0 - \tau), and \;\, for\, \dot{x}(t_0 + \varepsilon)$ So, in order for the initial value problem to make sense, one needs to give an initial function the value of $x(t)$ for the interval $[-r, 0]$. Each such initial function determines a unique solution to the delay differential equation. If we require that the initial functions to be continuous, then the solution space has the same dimensionality as $C([t_0 - \tau, t_0], \Re)$ [12,13].

The indefinite dimensional nature of delay differential equation is apparent in the study of linear systems. Just as for ordinary differential equations, one seeks exponential solutions and computes the characteristic equation.

2 Theoretical Concepts

Delay differential equations (DDE) provide a mathematical model for physical systems in which the rule of change of the systems depends not only on their present state, but also on their past history (state).

Consider the DDE below;

$$\dot{x}(t) - q(t)x(t-r) - dx(t) \qquad \text{…(2.1)}$$

$$where\ q(t) \le d,\ d > 0$$

In practice $q(t)$ will represent the nonlinearities of the equation. To understand the behaviour of the system, we compare its dynamics with those of the system

$$\dot{y}(t) = dy(t-\tau) - dy(t) \qquad \text{…(2.2)}$$

Lemma. If x and y are defined as above, and

$$x(t) = y(t) \geq 0,\ for\, t \in [e, e+t] \text{ for some } e, then\, x(t) \leq y(t), \quad \forall t.$$

[14].Ladas et al in 1983 has established a necessary and sufficient condition under which all solutions of the retarded differential equation

$$\dot{x}(t) + \sum_{i=1}^{m} qix(t-\tau_i) = 0,\ t \geq 0 \quad \text{oscillates} \qquad \text{…(2.3)}$$

where q_i are positive numbers

and τ_i are non-negative numbers, i=1,2,…,m

3 Application of DDE

The study of population dynamics in differential equation for single species population will be well established. The most common is the exponential and logistic growth models. This class of differential equation models will involve a time delay part.
Consider the model below;

$$x(t) = b(x(t-\tau))\dot{x}(t-\tau) - d(x(t))x(t) \qquad \text{…(3.1)}$$

Where b is non-increasing and d non-decreasing, which represents the population dynamics of a single species with a delayed birth term. The basic properties of this model are the type of functions b and d which might lead to the existence of periodic solutions in equation (3.1).We specify to use of the case $b(t) = be^{-at}\ and\ d(t)$ constant. It will prove the existence of the periodic solution of the equation.

The delay-dependent term is added to the parameter b and the effects of this alteration are explored, and conditions are given for the existence of linear instability of the positive steady state.

The model can be transformed into

$$x(t) = [b(x(t-\dot{\tau})) - d(x(t))]x(t) \qquad \text{…(3.2)}$$

The forms of $b(x)\, and\ d(x)$ might give rise to periodic solutions.

Consider that $b(x)$ is continuous, decreasing function that is the per capital growth rate of the population decreases with increased population levels. The delay can represent a gestation or maturation period, so the number of individuals entering the population depends on the levels of the population at the previous instance of time. The function $d(x)$ is non-decreasing and positive. This represents the per capita death rate, which may be increased by intra specific competition [12,15,16,14].

The type of these models has been extensively used in the mathematical biology literature, especially when there is an interest in modeling and oscillation phenomena. In population biology [16, 14] explored the model generally, while [17] is a specific application to housefly population. Oscillatory phenomena have few analytic results about the existence of periodic solutions [15].

Theorem: Let b and d be positive functions. Suppose that there exists $\bar{x}$ such that

$$sign(b(x)-d(x)) = -\mathrm{s}\,i\,\mathrm{gn}(x-\bar{x}), and \quad \forall x < d^{'}(x) \qquad \ldots(3.4)$$

Then $\bar{x}$ is a positive steady state, and the trivial steady state is unstable. If $b^{'}(\bar{x})\bar{x} > -2d(\bar{x}) - d^{'}(\bar{x})\bar{x},\ then\,\bar{x}$ is linearly stable for all x. Otherwise, there exists $\alpha\tau_c > 0\, such\ \ that\ \bar{x}$ is stable for $\tau < \tau_c$, and unstable for $\tau > \tau_c$

Proof

To begin with, $\bar{x}$ is a unique positive steady state, since $b(\bar{x}) - d(\bar{x}) = 0$ if and only if $x = \bar{x}$. It is the point at which $b(\bar{x}) = d(\bar{x})$. Line razing about the steady state yields the equation

$$\dot{x}(t) = (d(\bar{x}) + b^{'}(\bar{x})\bar{x})x(t-\tau) - (d(\bar{x}) + d^{'}(\bar{x})\bar{x})x(t)$$

which has the characteristic equation

$$\lambda = \alpha x(t-\tau) - \beta x(t),$$
$$where\ \ \alpha = d(\bar{x}) + b^{'}(\bar{x})\bar{x}\ and\ d(\bar{x}) + d^{'}(\bar{x})\bar{x}$$

since $b^{'}(\bar{x}) < d^{'}(\bar{x}),\ \alpha < \beta$.

Furthermore, we know that for $|\alpha| - |\beta| = \beta,$

all roots of the characteristic equation have negative real part.

Since $\alpha < \beta,$ then this condition is satisfied if and only if $\alpha > -\beta,$ but this is exactly the condition of the above equation (3.4).

If this is not the case, then $\alpha < -\beta$. It is clear that for i=0, the characteristic root is $\lambda = \alpha - \beta \le 0$. Thus, by the

3 .1 Application of DDE on Measles

We employed the basic stochastic model for epidemic processes. This is the continuous- infection type in which a susceptible becomes infectious immediately after the receipt of infection and continues in this state until removal from circulation by death or isolation. If the time elapsing between receiving infectious material and the development of infectiousness were short, and if the infectious period up to removal were approximately negative exponential in distribution, then such a model would be quite appropriate. Whether this closely mimics any actual disease is still uncertain, though scarlet fever and diphtheria. For example, the disease like measles at any rate, one version of this representation has met with some success, the small amount of data available passing the appropriate goodness-of-fit tests [6]. The data for the model were collected at Federal Medical Centre Gombe.

4 Method

The method used in solving the delay differential equation is the use of MATLAB. The delay differential equation is solved to its lowest level before applying MATLAB to generate results which are positive at all time, bounded and steady and stable. The graphs below explained the results of the delay differential equation (4.1).

Consider the equation

$$y'(t) = -\lambda y(t-1)(1+y(t))$$
$$for\ t \in [0,4], y(t) = t$$
$$\lambda = 1.5, 2, 2.5, 3$$

$$(\frac{\partial}{\partial t} y(t)) + \lambda y(t)(1+y(t))$$

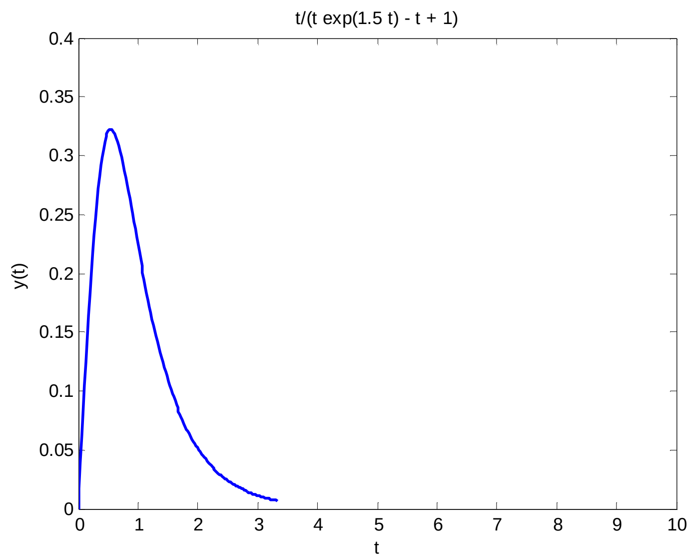
$$y(t) = \frac{1}{-1+\ell^{(\lambda t)} - C} \qquad \ldots(4.1)$$

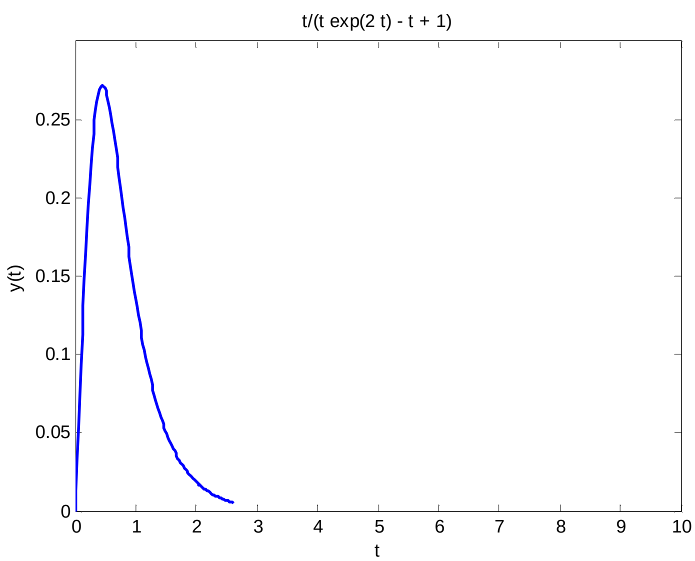
$$y(t) = \frac{t}{t\ell^{(\lambda t)} - t + 1}$$

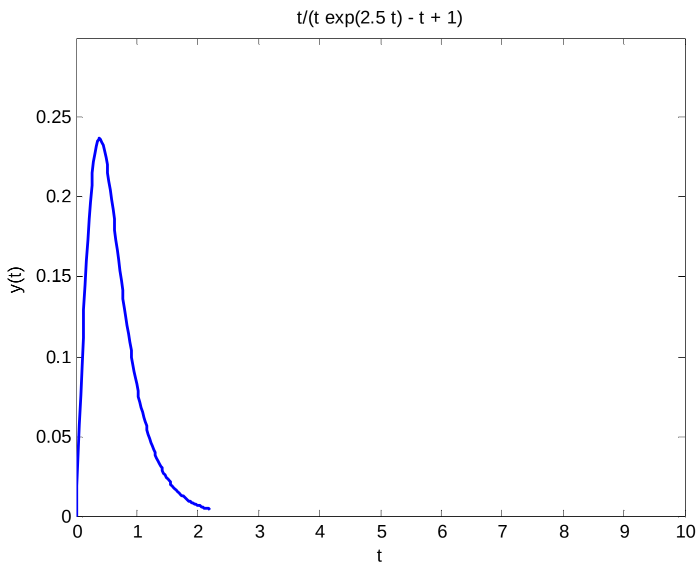
5 Results and Analysis

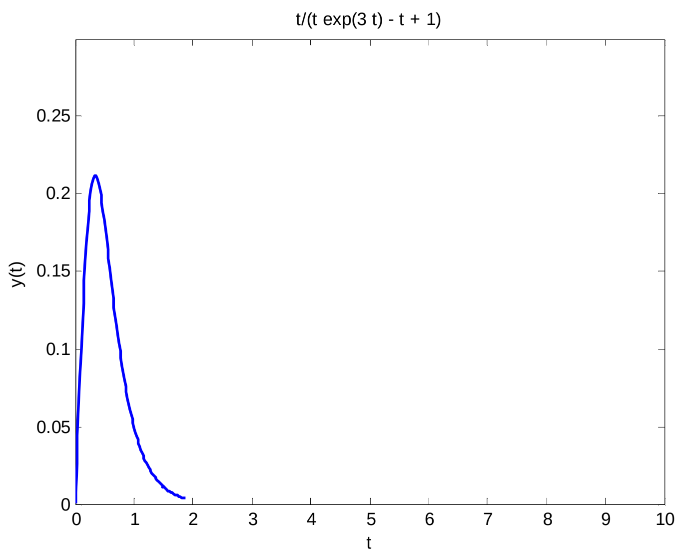
When the model is ran using MATLAB, it produced graphical view that varied from the parameter values, λ = 1.5, 2, 2.5 and 3. It will be realised that as the value of the parameter is smaller the curve is clearer and has longer times to cover than when the parameter value is larger. This also agreed with the pattern of the model and treatment is faster and more feasible at the early stages of the disease than when the disease must have stayed in the system. The results are displayed graphical forms below:

λ = 1.5


t/(t exp(1.5 t) - t + 1)
y(t)
t
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0
1
2
3
4
5
6
7
8
9
10


t/(t exp(2 t) - t + 1)
y(t)
t
0
0.05
0.1
0.15
0.2
0.25
0
1
2
3
4
5
6
7
8
9
10





6. Conclusion

The numerical scheme described above investigates the kind of time delay involved and because the time delay governed the dynamics of the system. The model with the time delay embedded the technique used to visualize the time series generated by the numerical integration of the model. We see as the values of the parameter, λ, are varied from 0 to 3, the curve changes pattern, the peak of the curve also changes gradually from 0.32 to 0.22 (the values of $y(t)$).

Competing Interests

Author has declared that no competing interests exist.

References

[1] Ciupe SM, BL de Bivort, Bortz DM, Nelson PW. Estimates of kinetic parameters from HIV patient data during primary infection through the eyes of three different models. Math. Biosci. In press; 2004.

[2] Nelso PW, Murrey JD, Perelson AS. A model of HIV-1 pathogenesis that includes an interaction delay. Math. Biosci. 2000;163:201-215.

[3] Cooke KK. Kuang, B Li. Analysis of an antiviral immune response model with time delays. Canad. Appl. Math. Quart. 1998;6(4):321-354.

[4] Zhao T. Global periodic solutions for a differential delay system modeling a microbial population in the chemo stat. J. Math. Anal. Appl. 1995;193:329-352.

[5] Smolen PD. Baxter, J Byrne. A reduced model clarifies the role of feedback loops and time delays in the Drosophila circadian ascillator. Biophys. J. 2002;83:2349-2359.

[6] Cooke KL, P van den Driessche, Zou X. Interaction of maturation delay and nonlinear birth in population and epidemic models. J. Math. Biol. 1999;39:332-354.

[7] Vielle B, G Chauvet. Delay equation analysis of human respiratory stability. Math. Biol, 1998;47(2):105-122.

[8] Villsana M, Radunskaya A. A delay differential equation model for tumor growth. J. Math. Biol. 2003;47(3):270-294.

[9] Comphell SA, Ewards B, P van de Driessche. Delayed coupling between two neural network loops. SIAM J. Appl. Math. 2004;65(1):316-335.

[10] Turchin P. Rarity of density dependence or population regulation with lags. Nature. 1990;344:660-663.

[11] Turchin P, Taylor AD. Complex dynamics in ecology time series. Ecology. 1992;73:289-305.

[12] Edelstein-Keshet L. Mathematical Models in Biology. McGraw-Hill, New York; 1988.

[13] El'sgol'ts LE, Norkin SB. An introduction to the theory and application of differential equations with deviating arguments. Academic Press, New York; 1973.

[14] Wangersky PJ, Cunningham WJ. On time lags in equations of growth. Proc. Nat. Acad. Sci. USA. 1956;42:699-702.

[15] Kuang K. Delay Differential equations with application to population biology. Academic Press, New York; 1993.

[16] Blythe SP. Instability and complex dynamic behaviour in population models with long time delays. Theor. Pop. Biol. 1982;22:147-176.

[17] Taylor CE, Sokal AD. Oscillation in housefly populations due to time lags. Ecology. 1976;57:1060-1067.

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