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Interval-valued Hesitant Multiplicative Preference Relations and Their Application to Multi-criteria Decision Making

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Abstract

Aims: The aim of this paper is to investigate interval-valued hesitant multiplicative preference relations and their application to multi-criteria decision making.
Study Design: Based on pseudo-multiplication, we define some basic operations for the interval-valued hesitant multiplicative sets (IVHMSs) and develop several aggregation operators for aggregating the interval-valued hesitant multiplicative information. Some desired properties and special cases of the developed operators are also investigated. Furthermore, we present a new preference structure named as the interval-valued hesitant multiplicative preference relation (IVHMPR), each element of which is an IVHMS, denoting all the possible interval multiplicative preference values offered by the decision makers for a paired comparison of alternatives.
Place and Duration of Study: Interval-valued hesitant fuzzy set (IVHFS), recently introduced by Chen et al., permits the membership degree of an element to a set to be represented as several possible interval values. However, it is noted that IVHFS uses 0.1–0.9 scale, which is inconsistent with some practical problems (e.g. the law of diminishing marginal utility in economics).
Methodology: We use the unsymmetrical 1–9 scale instead of the symmetrical 0.1–0.9 scale to express the membership degree information in the IVHFS and introduce the concept of interval-valued hesitant multiplicative set (IVHMS).
Results: An approach for multi-criteria decision making based on the interval-valued hesitant multiplicative preference relations (IVHMPRs) is developed and some numerical examples are provided to illustrate the developed approach.
Conclusion: We compare the IVHMPR with the interval-valued hesitant preference relation (IVHPR) and the interval multiplicative preference relation (IMPR), and show the effectiveness and practicality of the IVHMPR.

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1 Introduction

Hesitant fuzzy set (HFS), originally proposed by Torra [1], is an efficient tool for representing situations in which people hesitate between several numerical values to define the membership degree of an element to a set. Compared to some classical extensions of fuzzy set, such as interval-valued fuzzy set [2], intuitionistic fuzzy set [3], interval-valued intuitionistic fuzzy set [4] and type-2 fuzzy set [5], HFS can depict the human's hesitation more objectively and precisely. Since its introduction, HFS has attracted increasing interest in different areas and has been successfully applied to many practical fields, especially in decision making [6–23]. However, it should be noted that hesitant fuzzy set permits the membership of an element to be a set of several possible values. All these possible values are crisp real numbers that belong to $[0,1]$. However, in a lot of cases, no objective procedure is available for people to select the crisp membership degrees of elements in a set. It is suggested to specify an interval-valued membership degree to each element of the universe. To deal with such cases, Chen et al. [24] introduced the notion of interval-valued hesitant fuzzy set (IVHFS), which generalizes the HFS and permits the membership degree of an element to be a set of several possible interval values. The IVHFS can incorporate all possible opinions of the group members and, correspondingly, provides an intuitive description on the differences among the group members [24].

It is noted that both HFS and IVHFS use the balanced scale, i.e., 0.1–0.9 scale, to express the membership degree information. The 0.1–0.9 scale is a symmetrical distribution around 0.5 and assumes that the grades between "Extremely preferred" and "Extremely not preferred" are distributed uniformly and symmetrically, but in real life, the information are often asymmetrically distributed [25,26]. Take the law of diminishing marginal utility in economics as an example [27,28]. When given the same resources, a company with bad performance enhances more quickly than a company with good performance; that is, the gap between the grades expressing bad information should be smaller than the one between the grades expressing good information [27,28]. As an asymmetrically distributed scale, Saaty's 1–9 scale is more appropriate to deal with such a situation than the 0.1–0.9 scale (see Table 1 for more details [27,28]), motivated by which, we use the 1–9 scale instead of the 0.1–0.9 scale to express the membership degree information in the HFS and the IVHFS, and develop the concept of interval-valued hesitant multiplicative set (IVHMS), which permits the membership of an element to be a set of several possible interval multiplicative values. Then, based on pseudo-multiplication, we give some operational laws for the interval-valued hesitant multiplicative sets (IVHMSs), based on which, we further develop some interval-valued hesitant multiplicative aggregation operators, which can overcome the limitations of the interval-valued hesitant fuzzy aggregation operators [24].

Table 1. The comparison between the 0.1-0.9 scale and the 1-9 scale

1-9 scale	0.1-0.9 scale	Meaning
$1/9$	0.1	Extremely not preferred
$1/7$	0.2	Very strongly not preferred
$1/5$	0.3	Strongly not preferred
$1/3$	0.4	Moderately not preferred
1	0.5	Equally preferred
3	0.6	Moderately preferred
5	0.7	Strongly preferred
7	0.8	Very strongly preferred
9	0.9	Extremely preferred
other values between $1/9$ and 9	other values between 0 and 1	Intermediate values used to present compromise

In a group decision making (GDM) problem, preference relations are a powerful tool to describe the decision makers' preference information when they perform a paired comparison of alternatives. There are some different formats of preference relations, such as multiplicative preference relations [29], fuzzy preference relations [30], interval fuzzy preference relations [31], interval multiplicative preference relations [32], intuitionistic fuzzy preference relations [33,34], intuitionistic multiplicative preference relations [27,28], and so on. However, in some processes of decision making, due to time pressures and lack of knowledge, the decision makes (DMs) cannot provide their preference information with single numerical value, a margin of error or some possibility distribution values, but with several possible interval numbers. The aforementioned preference relations have difficulty in dealing with such situations. To solve this issue, Chen et al. [24] introduced the concept of interval-valued hesitant preference relation (IVHPR). Each element of the IVHPR is an interval-valued hesitant fuzzy element (IVHFE), which denotes all the possible interval preference values to which one alternative is preferred to another alternative. However, it is noted that the IVHPR uses the 0.1–0.9 scale to express the interval preference information. As mentioned before, the 0.1–0.9 scale has some disadvantages. Thus, we use the 1–9 scale instead of the 0.1–0.9 scale to describe the preference information in the IVHPR and define a new concept of interval-valued hesitant multiplicative preference relation (IVHMPR), each element of which is an interval-valued hesitant multiplicative element (IVHME) denoting all the possible interval multiplicative values to which one alternative is preferred to another alternative. Moreover, based on interval-valued hesitant multiplicative aggregation operators, we develop an approach to multi-criteria decision making with IVHMPRs and give some examples to illustrate the developed approach. Finally, we make a comparison analysis with the interval multiplicative preference relation (IMPR) and the IVHPR.

To do this, this paper is structured as follows. Section 2 recalls some concepts of hesitant fuzzy sets (HFSs) and interval-valued hesitant fuzzy sets (IVHFSs). In Section 3, we define the IVHMSs and give some operational laws for them. Section 4 presents several aggregation operators for interval-valued hesitant multiplicative information and examines some properties of the new operators. Section 5 develops an approach to multi-criteria decision making based on the IVHMPRs. In the sequel, the application of the developed approach is shown in Section 6. Some comparison analysis with the IVHPR and IMPR are also made in this section. The final section offers some concluding remarks.

2. Preliminaries

In this section, some basic concepts of hesitant fuzzy sets [1] and interval-valued hesitant fuzzy sets [24] are briefly introduced.

Definition 2.1 [1]. Let X be a fixed set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0,1]$.

To be easily understood, we express the HFS by a mathematical symbol:

$$E=\left\{\left\langle x,h_E(x)\right\rangle \middle| x\in X\right\} \tag{1}$$

where $h_E(x)$ is a set of some values in $[0,1]$, which denotes the possible membership degrees of the element $x\in X$ to the set E. For convenience, Xia and Xu [35] called $h=h_E(x)$ a hesitant fuzzy element (HFE) and H the set of all HFEs.

Considering that the precise membership degrees of an element to a set are sometimes hard to be specified, Chen et al. [24] proposed the concept of interval-valued hesitant fuzzy set, which permits the membership of an element to be a set of several possible interval values.

Definition 2.2 [24]. Let X be a reference set, and $D([0,1])$ be the set of all closed subintervals of $[0,1]$. An interval-valued hesitant fuzzy set (IVHFS) on X is expressed by

$$\tilde{A}=\left\{\left\langle x,\tilde{h}_{\tilde{A}}(x)\right\rangle \middle| x\in X\right\} \tag{2}$$

where $\tilde{h}_{\tilde{A}}(x)$ is a set of some interval values in $D([0,1])$, denoting all possible interval-valued membership degrees of the element $x\in X$ to the set $\tilde{A}$. For convenience, Chen et al. [24] called $\tilde{h}=\tilde{h}_{\tilde{A}}(x)$ an interval-valued hesitant fuzzy element (IVHFE) and $\tilde{H}$ the set of all IVHFEs. If $\tilde{\gamma}\in\tilde{h}$, then $\tilde{\gamma}$ is an interval and it can be denoted by $\tilde{\gamma}=\left[\tilde{\gamma}^L,\tilde{\gamma}^U\right]$.

Definition 2.3 [24]. Let $X=\{x_1,x_2,\cdots,x_n\}$ be a fixed set. An interval-valued hesitant preference relation (IVHPR) on X is denoted by a matrix $\tilde{R}=\left(\tilde{r}_{ij}\right)_{n\times n}\subset X\times X$, where $\tilde{r}_{ij}=\left\{\tilde{r}_{ij}^s \middle| s=1,2,\cdots,l_{\tilde{r}_{ij}}\right\}$ is an IVHFE, indicating all possible degrees to which x_i is preferred to x_j, and $l_{\tilde{r}_{ij}}$ represents the number of intervals in an IVHFE. Moreover, $\tilde{r}_{ij}$ should satisfy

$$\inf \tilde{r}_{ij}^{\sigma(s)}+\sup \tilde{r}_{ji}^{\sigma\left(l_{\tilde{r}_{ij}}-s+1\right)}=\sup \tilde{r}_{ij}^{\sigma(s)}+\inf \tilde{r}_{ji}^{\sigma\left(l_{\tilde{r}_{ij}}-s+1\right)}=1,\ \tilde{r}_{ii}=\{[1,1]\},\ \forall i,j=1,2,\cdots,n \tag{3}$$

where the elements in $\tilde{r}_{ij}$ are arranged in an increasing order, $\tilde{r}_{ij}^{\sigma(s)}$ denotes the *s*th smallest value in $\tilde{r}_{ij}$, and $\inf \tilde{r}_{ij}^{\sigma(s)}$ and $\sup \tilde{r}_{ij}^{\sigma(s)}$ denote the lower and upper limits of $\tilde{r}_{ij}^{\sigma(s)}$, respectively.

Definition 2.4 [36]. The pseudo-multiplication $\odot$ is defined as: $a \odot b = g^{-1}(g(a)\cdot g(b))$, where g is a strictly decreasing function such that $g:(0,\infty)\to(0,\infty)$.

3. Interval-valued Hesitant Multiplicative Sets (IVHMSS)

Chen et al. [24] proposed some aggregation operators for aggregating interval-valued hesitant fuzzy information. Among them, the interval-valued hesitant fuzzy weighted averaging (IVHFWA) operator is the basic one, based on which, other aggregation operators have been developed.

Definition 3.1 [24]. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHFEs, and let $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $\tilde{h}_i$ ($i=1,2,\cdots,n$) with $w_i\in[0,1]$ and $\sum_{i=1}^{n} w_i = 1$. An interval-valued hesitant fuzzy weighted averaging (IVHFWA) operator is a mapping $\tilde{H}^n \to \tilde{H}$ such that

$$\text{IVHFWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigoplus_{i=1}^{n}\left(w_i\tilde{h}_i\right)=\left\{\left[1-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_i^L\right)^{w_i},1-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_i^U\right)^{w_i}\right]\middle|\tilde{\gamma}_1\in\tilde{h}_1,\tilde{\gamma}_2\in\tilde{h}_2,\cdots,\tilde{\gamma}_n\in\tilde{h}_n\right\} \quad (4)$$

However, the IVHFWA operator has a drawback that is shown as follows:

Example 3.1. Let $\tilde{h}_1=\{[1,1]\}$, $\tilde{h}_2=\{[0,0]\}$, $\tilde{h}_3=\{[0,0]\}$, $\tilde{h}_4=\{[0,0]\}$, $\tilde{h}_5=\{[0,0]\}$, and $\tilde{h}_6=\{[0,0]\}$ be six special IVHFEs, and $w=\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)^T$ be their weight vector, then by the IVHFWA operator, we have

$$\text{IVHFWA}\left(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3,\tilde{h}_4,\tilde{h}_5,\tilde{h}_6\right)=\{[1,1]\}$$

which is somewhat inconsistent with our intuition. In the following, we will try to address this issue by using other scales to express the interval-valued hesitant fuzzy information.

In Definition 3.1, if we use the 1-9 scale instead of the 0.1-0.9 scale to express the membership degree, then a new concept can be introduced as below:

Definition 3.2. Let X be a reference set. A interval-valued hesitant multiplicative set (IVHMS) on X is defined as

$$\bar{M}=\left\{\left\langle x,\bar{h}_{\bar{M}}(x)\right\rangle\middle| x\in X\right\} \quad (5)$$

where $\bar{h}_{\bar{M}}(x)=\{\bar{\gamma}|\bar{\gamma}=[\bar{\gamma}^L,\bar{\gamma}^U]\in \bar{h}_{\bar{M}}(x)\}$ denotes all possible interval-valued membership degrees of the element $x\in X$ to the set $\bar{M}$, with the condition:

$$\frac{1}{9}\le \bar{\gamma}^L \le \bar{\gamma}^U \le 9\,,\quad \forall x\in X\,,\ \forall\bar{\gamma}=[\bar{\gamma}^L,\bar{\gamma}^U]\in \bar{h}_{\bar{M}}(x) \tag{6}$$

For convenience, we call $\bar{h}=\bar{h}_{\bar{M}}(x)$ an interval-valued hesitant multiplicative element (IVHME) and $\bar{H}$ the set of all interval-valued hesitant multiplicative elements (IVHMEs).

Example 3.2. Let $X=\{x_1,x_2,x_3\}$,
$\bar{M}=\left\{\left\langle x_1,\left\{\left[\frac{1}{8},\frac{1}{6}\right],\left[\frac{1}{4},\frac{1}{3}\right]\right\}\right\rangle,\left\langle x_2,\left\{\left[\frac{1}{2},1\right],[1,3],[2,3]\right\}\right\rangle,\left\langle x_3,\{[6,7],[8,9]\}\right\rangle\right\}$, and
$\bar{h}=\left\{\left[\frac{1}{2},1\right],[1,3],[2,3]\right\}$. Then, $\bar{M}$ is an IVHMS on X, and $\bar{h}$ is an IVHME.

Given three IVHMEs expressed by $\bar{h}$, $\bar{h}_1$, and $\bar{h}_2$, we define some basic operations on them as below:

$$\text{(1) } \bar{h}^c=\{[1/\bar{\gamma}^U,1/\bar{\gamma}^L]\,|\,\bar{\gamma}\in\bar{h}\}; \tag{7}$$

$$\text{(2) } \bar{h}_1\cup\bar{h}_2=\{[\bar{\gamma}_1^L\vee\bar{\gamma}_2^L,\bar{\gamma}_1^U\vee\bar{\gamma}_2^U]\,|\,\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2\}; \tag{8}$$

$$\text{(3) } \bar{h}_1\cap\bar{h}_2=\{[\bar{\gamma}_1^L\wedge\bar{\gamma}_2^L,\bar{\gamma}_1^U\wedge\bar{\gamma}_2^U]\,|\,\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2\}. \tag{9}$$

To compare the IVHMEs, we define the following comparison laws:

Definition 3.3. For an IVHME $\bar{h}=\{[\bar{\gamma}^L,\bar{\gamma}^U]\,|\,\bar{\gamma}\in\bar{h}\}$, $s(\bar{h})=\left(\prod_{\bar{\gamma}\in\bar{h}}(\bar{\gamma}^L\cdot\bar{\gamma}^U)\right)^{\frac{1}{2l_{\bar{h}}}}$ is called the score function of $\bar{h}$, where $l_{\bar{h}}$ is the number of the elements in $\bar{h}$. For two IVHMEs $\bar{h}_1$ and $\bar{h}_2$, if $s(\bar{h}_1)>s(\bar{h}_2)$, then $\bar{h}_1>\bar{h}_2$; if $s(\bar{h}_1)=s(\bar{h}_2)$, then $\bar{h}_1=\bar{h}_2$.

Based on the pseudo-multiplication, motivated by the work of Xia and Xu [15], we further define some operations about IVHMEs as below:

$$\text{(1) } \bar{h}_1\oplus\bar{h}_2=\{[f^{-1}(f(\bar{\gamma}_1^L)\cdot f(\bar{\gamma}_2^L)),f^{-1}(f(\bar{\gamma}_1^U)\cdot f(\bar{\gamma}_2^U))]\,|\,\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2\}; \tag{10}$$

$$\text{(2) } \bar{h}_1\otimes\bar{h}_2=\{[g^{-1}(g(\bar{\gamma}_1^L)\cdot g(\bar{\gamma}_2^L)),g^{-1}(g(\bar{\gamma}_1^U)\cdot g(\bar{\gamma}_2^U))]\,|\,\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2\}; \tag{11}$$

$$\text{(3) } \lambda\bar{h}=\left\{\left[f^{-1}\left((f(\bar{\gamma}^L))^{\lambda}\right),f^{-1}\left((f(\bar{\gamma}^U))^{\lambda}\right)\right]\,|\,\bar{\gamma}\in\bar{h}\right\},\ \lambda>0; \tag{12}$$

(4) $\bar{h}^{\lambda} = \left\{ \left[g^{-1}\left(\left(g\left(\bar{\gamma}^{L} \right) \right)^{\lambda} \right), g^{-1}\left(\left(g\left(\bar{\gamma}^{U} \right) \right)^{\lambda} \right) \right] \middle| \bar{\gamma} \in \bar{h} \right\}, \ \lambda > 0$. (13)

where g is a strictly decreasing function such that $g:(0,\infty) \to (0,\infty)$, and $f(t) = g\left(\frac{1}{t}\right)$.

Theorem 3.1. For three IVHMEs $\bar{h}$, $\bar{h}_1$, and $\bar{h}_2$, we have the following properties:

(1) $\bar{h}_1 \oplus \bar{h}_2 = \bar{h}_2 \oplus \bar{h}_1$;
(2) $\bar{h}_1 \otimes \bar{h}_2 = \bar{h}_2 \otimes \bar{h}_1$;
(3) $\lambda\left(\bar{h}_1 \oplus \bar{h}_2\right) = \lambda \bar{h}_1 \oplus \lambda \bar{h}_2, \ \ \lambda > 0$;
(4) $\left(\bar{h}_1 \otimes \bar{h}_2\right)^{\lambda} = \bar{h}_1^{\lambda} \otimes \bar{h}_2^{\lambda}, \ \ \lambda > 0$;
(5) $\lambda_1 \bar{h} \oplus \lambda_2 \bar{h} = (\lambda_1 + \lambda_2)\bar{h}, \ \ \lambda_1, \lambda_2 > 0$;
(6) $\bar{h}^{\lambda_1} \otimes \bar{h}^{\lambda_2} = \bar{h}^{\lambda_1 + \lambda_2}, \ \ \lambda_1, \lambda_2 > 0$.

Proof. It can be easily derived from Eqs. (10), (11), (12) and (13).

4. Aggregation Operators for Interval-valued Hesitant Multiplicative Information

In the current section, we will propose several operators for aggregating the interval-valued hesitant multiplicative information and investigate some properties and special cases of these operators.

4.1. The GIVHMWA and IVHMWA Operators

Definition 4.1. Let $\bar{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\bar{h}_i$ ($i = 1, 2, \cdots, n$) with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, a generalized interval-valued hesitant multiplicative weighted averaging (GIVHMWA) operator is a mapping $\bar{H}^n \to \bar{H}$, where

$$\text{GIVHMWA}_{\lambda}\left(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_n\right) = \left(\mathop{\oplus}_{i=1}^{n} \left(w_i \bar{h}_i^{\lambda} \right) \right)^{1/\lambda} \tag{14}$$

with $\lambda > 0$.

Especially, if $\lambda = 1$, then the GIVHMWA operator reduces to the interval-valued hesitant multiplicative weighted averaging (IVHMWA) operator:

$$\text{IVHMWA}\left(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_n\right) = \mathop{\oplus}_{i=1}^{n} \left(w_i \bar{h}_i \right) \tag{15}$$

Theorem 4.1. Let $\overline{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\overline{h}_i$ ($i = 1, 2, \cdots, n$), where w_i indicates the importance degree of $\overline{h}_i$, satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by using the GIVHMWA operator is also an IVHME, and

$$\mathrm{GIVHMWA}_\lambda\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right) = \left\{ \left[\begin{array}{l} g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\overline{\gamma}_i^L\right)\right)^\lambda\right)\right)\right)^{w_i}\right)\right)\right)^{1/\lambda}\right), \\ g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\overline{\gamma}_i^U\right)\right)^\lambda\right)\right)\right)^{w_i}\right)\right)\right)^{1/\lambda}\right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\} \tag{16}$$

Proof. First, we will prove the following equation:

$$\mathop{\oplus}_{i=1}^{n}\left(w_i \overline{h}_i^\lambda\right) = \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\overline{\gamma}_i^L\right)\right)^\lambda\right)\right)\right)^{w_i}\right), \\ f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\overline{\gamma}_i^U\right)\right)^\lambda\right)\right)\right)^{w_i}\right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\} \tag{17}$$

By using mathematical induction on n: For $n = 2$, since

$$w_1 \overline{h}_1^\lambda = \left\{ \left[f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\overline{\gamma}_1^L\right)\right)^\lambda\right)\right)\right)^{w_1}\right), f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\overline{\gamma}_1^U\right)\right)^\lambda\right)\right)\right)^{w_1}\right) \right] \middle| \overline{\gamma}_1 \in \overline{h}_1 \right\}$$

$$w_2 \overline{h}_2^\lambda = \left\{ \left[f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\overline{\gamma}_2^L\right)\right)^\lambda\right)\right)\right)^{w_2}\right), f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\overline{\gamma}_2^U\right)\right)^\lambda\right)\right)\right)^{w_2}\right) \right] \middle| \overline{\gamma}_2 \in \overline{h}_2 \right\}$$

we have

$$
\begin{aligned}
& w_1\bar{h}_1^{\lambda} \oplus w_2\bar{h}_2^{\lambda} \\
&= \left(\left\{ \left[f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_1^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_1} \right), f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_1^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_1} \right) \right] \Big| \bar{\gamma}_1 \in \bar{h}_1 \right\} \right) \\
&\qquad \oplus \left(\left\{ \left[f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_2^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_2} \right), f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_2^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_2} \right) \right] \Big| \bar{\gamma}_2 \in \bar{h}_2 \right\} \right) \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(f\left(f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_1^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_1} \right) \right) \cdot f\left(f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_2^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_2} \right) \right) \right), \\ f^{-1}\left(f\left(f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_1^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_1} \right) \right) \cdot f\left(f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_2^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_2} \right) \right) \right) \end{array} \right] \Bigg| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2 \right\} \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_1^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_1} \cdot \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_2^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_2} \right), \\ f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_1^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_1} \cdot \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_2^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_2} \right) \end{array} \right] \Bigg| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2 \right\}
\end{aligned}
$$

That is, the Eq. (17) holds for $n = 2$. Suppose that the Eq. (17) holds for $n = k$, i.e.,

$$
\overset{k}{\underset{i=1}{\oplus}} \left(w_i \bar{h}_i \right) = \left\{ \left[f^{-1}\left(\prod_{i=1}^{k} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \right), f^{-1}\left(\prod_{i=1}^{k} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \right) \right] \Big| \bar{\gamma}_1 \in \bar{h}_1, \cdots, \bar{\gamma}_k \in \bar{h}_k \right\}
$$

then, when $n = k + 1$, we have

$$
\begin{aligned}
& \overset{k+1}{\underset{i=1}{\oplus}} \left(w_i \bar{h}_i \right) \\
&= \left(\overset{k}{\underset{i=1}{\oplus}} \left(w_i \bar{h}_i \right) \right) \oplus \left(w_{k+1} \bar{h}_{k+1} \right) \\
&= \left(\left\{ \left[f^{-1}\left(\prod_{i=1}^{k} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \right), f^{-1}\left(\prod_{i=1}^{k} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \right) \right] \Big| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_k \in \bar{h}_k \right\} \right) \\
&\qquad \oplus \left(\left\{ \left[f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_{k+1}^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_{k+1}} \right), f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_{k+1}^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_{k+1}} \right) \right] \Big| \bar{\gamma}_{k+1} \in \bar{h}_{k+1} \right\} \right) \\
&= \left\{ \begin{array}{l} \left[\begin{array}{l} f^{-1}\left(f\left(f^{-1}\left(\prod_{i=1}^{k} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \right) \right) \cdot f\left(f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_{k+1}^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_{k+1}} \right) \right) \right), \\ f^{-1}\left(f\left(f^{-1}\left(\prod_{i=1}^{k} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \right) \right) \cdot f\left(f^{-1}\left(\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_{k+1}^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_{k+1}} \right) \right) \right) \end{array} \right] \\ \Big| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_k \in \bar{h}_k, \bar{\gamma}_{k+1} \in \bar{h}_{k+1} \end{array} \right\} \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{k} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \cdot \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_{k+1}^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_{k+1}} \right), \\ f^{-1}\left(\prod_{i=1}^{k} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \cdot \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_{k+1}^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_{k+1}} \right) \end{array} \right] \Bigg| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_k \in \bar{h}_k, \bar{\gamma}_{k+1} \in \bar{h}_{k+1} \right\} \\
&= \left\{ \left[f^{-1}\left(\prod_{i=1}^{k+1} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{L}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \right), f^{-1}\left(\prod_{i=1}^{k+1} \left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^{U}\right) \right)^{\lambda} \right) \right) \right)^{w_i} \right) \right] \Big| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_k \in \bar{h}_k, \bar{\gamma}_{k+1} \in \bar{h}_{k+1} \right\}
\end{aligned}
$$

i.e., Eq. (17) holds for $n = k + 1$. Thus Eq. (17) holds for all n.

Furthermore, by Eq. (13), we have

$$\mathrm{GIVHMWA}_\lambda\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left(\mathop{\oplus}_{i=1}^{n}\left(w_i\bar{h}_i^\lambda\right)\right)^{1/\lambda}$$

$$=\left(\left\{\left[f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^L\right)\right)^\lambda\right)\right)\right)^{w_i}\right),f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^U\right)\right)^\lambda\right)\right)\right)^{w_i}\right)\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\}\right)^{1/\lambda}$$

$$=\left\{\left[\begin{array}{l}g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^L\right)\right)^\lambda\right)\right)\right)^{w_i}\right)\right)\right)^{1/\lambda}\right),\\ g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^U\right)\right)^\lambda\right)\right)\right)^{w_i}\right)\right)\right)^{1/\lambda}\right)\end{array}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\}$$

In addition, because $g:(0,\infty)\to(0,\infty)$ is a strictly decreasing function and $f(t)=g(1/t)$, $f:(0,\infty)\to(0,\infty)$ is a strictly increasing function. Accordingly, $g^{-1}:(0,\infty)\to(0,\infty)$ is a strictly decreasing function and $f^{-1}:(0,\infty)\to(0,\infty)$ is a strictly increasing function. Moreover, for any $\bar{\gamma}_i=\left[\bar{\gamma}_i^L,\bar{\gamma}_i^U\right]\in\bar{h}_i$ ($i=1,2,\cdots,n$), we have $\frac{1}{9}\le\bar{\gamma}_i^L\le\bar{\gamma}_i^U\le 9$. Therefore,

$$\frac{1}{9}=g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\frac{1}{9}\right)\right)^\lambda\right)\right)\right)^{w_i}\right)\right)\right)^{1/\lambda}\right)\le g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^L\right)\right)^\lambda\right)\right)\right)^{w_i}\right)\right)\right)^{1/\lambda}\right)$$

$$\le g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^U\right)\right)^\lambda\right)\right)\right)^{w_i}\right)\right)\right)^{1/\lambda}\right)\le g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g(9)\right)^\lambda\right)\right)\right)^{w_i}\right)\right)\right)^{1/\lambda}\right)=9$$

This completes the proof of Theorem 4.1.

Based on Theorem 4.1, when $\lambda=1$, the following theorem can be easily obtained:

Theorem 4.2. Let $\bar{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHMEs, and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weight vector of $\bar{h}_i$ ($i=1,2,\cdots,n$), where w_i indicates the importance degree of $\bar{h}_i$, satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, then the aggregated value by using the IVHMWA operator is also an IVHME, and

$$\mathrm{IVHMWA}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left\{\left[f^{-1}\left(\prod_{i=1}^{n}\left(f\left(\bar{\gamma}_i^L\right)\right)^{w_i}\right),f^{-1}\left(\prod_{i=1}^{n}\left(f\left(\bar{\gamma}_i^U\right)\right)^{w_i}\right)\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (18)$$

Especially, if $w=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$, then the GIVHMWA operator reduces to the generalized interval-valued hesitant multiplicative averaging (GIVHMA) operator:

$$\text{GIVHMA}_\lambda\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left(\mathop{\oplus}_{i=1}^{n}\left(\frac{1}{n}\bar{h}_i^{\lambda}\right)\right)^{1/\lambda}$$

$$=\left\{\left[\begin{array}{l} g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^L\right)\right)^{\lambda}\right)\right)\right)^{\frac{1}{n}}\right)\right)\right)^{1/\lambda}\right), \\ g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(g^{-1}\left(\left(g\left(\bar{\gamma}_i^U\right)\right)^{\lambda}\right)\right)\right)^{\frac{1}{n}}\right)\right)\right)^{1/\lambda}\right) \end{array}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (19)$$

If $w=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$, then the IVHMWA operator reduces to the interval-valued hesitant multiplicative averaging (IVHMA) operator:

$$\text{IVHMA}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\mathop{\oplus}_{i=1}^{n}\left(\frac{1}{n}\bar{h}_i\right)$$

$$=\left\{\left[f^{-1}\left(\prod_{i=1}^{n}\left(f\left(\bar{\gamma}_i^L\right)\right)^{\frac{1}{n}}\right),f^{-1}\left(\prod_{i=1}^{n}\left(f\left(\bar{\gamma}_i^U\right)\right)^{\frac{1}{n}}\right)\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (20)$$

Then, we can investigate some desirable properties of the IVHMWA operator as follows:

Theorem 4.3. Let $\bar{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHMEs, and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weight vector of $\bar{h}_i$ ($i=1,2,\cdots,n$), satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, if $\bar{h}$ is an IVHME, then

$$\text{IVHMWA}\left(\bar{h}_1\oplus\bar{h},\bar{h}_2\oplus\bar{h},\cdots,\bar{h}_n\oplus\bar{h}\right)=\text{IVHMWA}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)\oplus\bar{h} \quad (21)$$

Proof. Based on Eq. (10), for any $i=1,2,\cdots,n$,

$$\bar{h}_i\oplus\bar{h}=\bar{h}_1\oplus\bar{h}_2=\left\{\left[f^{-1}\left(f\left(\bar{\gamma}_i^L\right)\cdot f\left(\bar{\gamma}^L\right)\right),f^{-1}\left(f\left(\bar{\gamma}_i^U\right)\cdot f\left(\bar{\gamma}^U\right)\right)\right]\middle|\bar{\gamma}_i\in\bar{h}_i,\bar{\gamma}\in\bar{h}\right\}$$

According to Theorem 4.2, we have

$$\text{IVHMWA}\left(\bar{h}_1 \oplus \bar{h}, \bar{h}_2 \oplus \bar{h}, \cdots, \bar{h}_n \oplus \bar{h}\right)$$

$$= \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{n} \left(f\left(f^{-1}\left(f\left(\bar{\gamma}_i^L\right) \cdot f\left(\bar{\gamma}^L\right) \right) \right) \right)^{w_i} \right), \\ f^{-1}\left(\prod_{i=1}^{n} \left(f\left(f^{-1}\left(f\left(\bar{\gamma}_i^U\right) \cdot f\left(\bar{\gamma}^U\right) \right) \right) \right)^{w_i} \right) \end{array} \right] \middle| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n, \bar{\gamma} \in \bar{h} \right\}$$

$$= \left\{ \left[f^{-1}\left(f\left(\bar{\gamma}^L\right) \cdot \prod_{i=1}^{n} \left(f\left(\bar{\gamma}_i^L\right) \right)^{w_i} \right), f^{-1}\left(f\left(\bar{\gamma}^U\right) \cdot \prod_{i=1}^{n} \left(f\left(\bar{\gamma}_i^U\right) \right)^{w_i} \right) \right] \middle| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n, \bar{\gamma} \in \bar{h} \right\}$$

By Theorem 4.2 and Eq. (10), we can obtain

$$\text{IVHMWA}\left(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_n\right) \oplus \bar{h}$$

$$= \left\{ \left[f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\bar{\gamma}_i^L\right) \right)^{w_i} \right), f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\bar{\gamma}_i^U\right) \right)^{w_i} \right) \right] \middle| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n \right\} \oplus \left\{ \left[\bar{\gamma}^L, \bar{\gamma}^U \right] \middle| \bar{\gamma} \in \bar{h} \right\}$$

$$= \left\{ \left[\begin{array}{l} f^{-1}\left(f\left(f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\bar{\gamma}_i^L\right) \right)^{w_i} \right) \right) \cdot f\left(\bar{\gamma}^L\right) \right), \\ f^{-1}\left(f\left(f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\bar{\gamma}_i^U\right) \right)^{w_i} \right) \right) \cdot f\left(\bar{\gamma}^U\right) \right) \end{array} \right] \middle| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n, \bar{\gamma} \in \bar{h} \right\}$$

$$= \left\{ \left[f^{-1}\left(f\left(\bar{\gamma}^L\right) \cdot \prod_{i=1}^{n} \left(f\left(\bar{\gamma}_i^L\right) \right)^{w_i} \right), f^{-1}\left(f\left(\bar{\gamma}^U\right) \cdot \prod_{i=1}^{n} \left(f\left(\bar{\gamma}_i^U\right) \right)^{w_i} \right) \right] \middle| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n, \bar{\gamma} \in \bar{h} \right\}$$

This completes the proof of Theorem 4.3.

Theorem 4.4. Let $\bar{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = \left(w_1, w_2, \cdots, w_n\right)^T$ be the weight vector of $\bar{h}_i$ ($i = 1, 2, \cdots, n$), satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, if $r > 0$, then

$$\text{IVHMWA}\left(r\bar{h}_1, r\bar{h}_2, \cdots, r\bar{h}_n\right) = r\text{IVHMWA}\left(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_n\right) \tag{22}$$

Proof. Since for any $i = 1, 2, \cdots, n$,

$$r\bar{h}_i = \left\{ \left[f^{-1}\left(\left(f\left(\bar{\gamma}_i^L\right) \right)^r \right), f^{-1}\left(\left(f\left(\bar{\gamma}_i^U\right) \right)^r \right) \right] \middle| \bar{\gamma}_i \in \bar{h}_i \right\}$$

Based on Theorem 4.2, we have

$$\text{IVHMWA}\left(r\bar{h}_1, r\bar{h}_2, \cdots, r\bar{h}_n\right)$$

$$= \left\{ \left[f^{-1}\left(\prod_{i=1}^{n} \left(f\left(f^{-1}\left(\left(f\left(\bar{\gamma}_i^L\right) \right)^r \right) \right) \right)^{w_i} \right), f^{-1}\left(\prod_{i=1}^{n} \left(f\left(f^{-1}\left(\left(f\left(\bar{\gamma}_i^U\right) \right)^r \right) \right) \right)^{w_i} \right) \right] \middle| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n \right\}$$

$$= \left\{ \left[f^{-1}\left(\prod_{i=1}^{n} \left(\left(f\left(\bar{\gamma}_i^L\right) \right)^r \right)^{w_i} \right), f^{-1}\left(\prod_{i=1}^{n} \left(\left(f\left(\bar{\gamma}_i^U\right) \right)^r \right)^{w_i} \right) \right] \middle| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n \right\}$$

According to Eq. (12), we can obtain

$$r\text{IVHMWA}\left(\overline{h}_1,\overline{h}_2,\cdots,\overline{h}_n\right)$$
$$=r\left(\left\{\left[f^{-1}\left(\prod_{i=1}^{n}\left(f\left(\overline{\gamma}_i^L\right)\right)^{w_i}\right),f^{-1}\left(\prod_{i=1}^{n}\left(f\left(\overline{\gamma}_i^U\right)\right)^{w_i}\right)\right]\middle|\overline{\gamma}_1\in\overline{h}_1,\overline{\gamma}_2\in\overline{h}_2,\cdots,\overline{\gamma}_n\in\overline{h}_n\right\}\right)$$
$$=\left\{\left[f^{-1}\left(\left(f\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(\overline{\gamma}_i^L\right)\right)^{w_i}\right)\right)\right)^r\right),f^{-1}\left(\left(f\left(f^{-1}\left(\prod_{i=1}^{n}\left(f\left(\overline{\gamma}_i^U\right)\right)^{w_i}\right)\right)\right)^r\right)\right]\middle|\overline{\gamma}_1\in\overline{h}_1,\overline{\gamma}_2\in\overline{h}_2,\cdots,\overline{\gamma}_n\in\overline{h}_n\right\}$$
$$=\left\{\left[f^{-1}\left(\prod_{i=1}^{n}\left(\left(f\left(\overline{\gamma}_i^L\right)\right)^r\right)^{w_i}\right),f^{-1}\left(\prod_{i=1}^{n}\left(\left(f\left(\overline{\gamma}_i^U\right)\right)^r\right)^{w_i}\right)\right]\middle|\overline{\gamma}_1\in\overline{h}_1,\overline{\gamma}_2\in\overline{h}_2,\cdots,\overline{\gamma}_n\in\overline{h}_n\right\}$$

This completes the proof of Theorem 4.4.

According to Theorems 4.3 and 4.4, we can easily obtain Theorem 4.5:

Theorem 4.5. Let $\overline{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHMEs, and $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the weight vector of $\overline{h}_i$ ($i=1,2,\cdots,n$), satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, if $r>0$ and $\overline{h}$ is an IVHME, then

$$\text{IVHMWA}\left(r\overline{h}_1\oplus\overline{h},r\overline{h}_2\oplus\overline{h},\cdots,r\overline{h}_n\oplus\overline{h}\right)=r\text{IVHMWA}\left(\overline{h}_1,\overline{h}_2,\cdots,\overline{h}_n\right)\oplus\overline{h} \tag{23}$$

Theorem 4.6. Let $\overline{h}_i$ and $\overline{l}_i$ ($i=1,2,\cdots,n$) be two collections of IVHMEs, $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be their weight vector with $w_i\in[0,1]$ ($i=1,2,\cdots,n$) and $\sum_{i=1}^{n}w_i=1$, then

$$\text{IVHMWA}\left(\overline{h}_1\oplus\overline{l}_1,\overline{h}_2\oplus\overline{l}_2,\cdots,\overline{h}_n\oplus\overline{l}_n\right)=\text{IVHMWA}\left(\overline{h}_1,\overline{h}_2,\cdots,\overline{h}_n\right)\oplus\text{IVHMWA}\left(\overline{l}_1,\overline{l}_2,\cdots,\overline{l}_n\right) \tag{24}$$

Proof. According to Eq. (10), we have

$$\overline{h}_i\oplus\overline{l}_i=\left\{\left[f^{-1}\left(f\left(\overline{\gamma}_i^L\right)\cdot f\left(\overline{\xi}_i^L\right)\right),f^{-1}\left(f\left(\overline{\gamma}_i^U\right)\cdot f\left(\overline{\xi}_i^U\right)\right)\right]\middle|\overline{\gamma}_i\in\overline{h}_i,\overline{\xi}_i\in\overline{l}_i\right\}$$

According to Theorem 4.2, we have

$$
\begin{aligned}
&\mathrm{IVHMWA}\left(\overline{h}_1 \oplus \overline{l}_1, \overline{h}_2 \oplus \overline{l}_2, \cdots, \overline{h}_n \oplus \overline{l}_n\right) \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{n} \left(f\left(f^{-1}\left(f\left(\overline{\gamma}_i^L\right) \cdot f\left(\overline{\xi}_i^L\right)\right)\right)\right)^{w_i} \right), \\ f^{-1}\left(\prod_{i=1}^{n} \left(f\left(f^{-1}\left(f\left(\overline{\gamma}_i^U\right) \cdot f\left(\overline{\xi}_i^U\right)\right)\right)\right)^{w_i} \right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \cdots, \overline{\gamma}_n \in \overline{h}_n, \overline{\xi}_1 \in \overline{l}_1, \cdots, \overline{\xi}_n \in \overline{l}_n \right\} \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^L\right) \cdot f\left(\overline{\xi}_i^L\right)\right)^{w_i} \right), \\ f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^U\right) \cdot f\left(\overline{\xi}_i^U\right)\right)^{w_i} \right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \cdots, \overline{\gamma}_n \in \overline{h}_n, \overline{\xi}_1 \in \overline{l}_1, \cdots, \overline{\xi}_n \in \overline{l}_n \right\}
\end{aligned}
$$

On the other hand, according to Theorem 4.2 and Eq. (10), we have

$$
\begin{aligned}
&\mathrm{IVHMWA}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right) \oplus \mathrm{IVHMWA}\left(\overline{l}_1, \overline{l}_2, \cdots, \overline{l}_n\right) \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^L\right)\right)^{w_i} \right), \\ f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^U\right)\right)^{w_i} \right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\} \oplus \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\xi}_i^L\right)\right)^{w_i} \right), \\ f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\xi}_i^U\right)\right)^{w_i} \right) \end{array} \right] \middle| \overline{\xi}_1 \in \overline{l}_1, \overline{\xi}_2 \in \overline{l}_2, \cdots, \overline{\xi}_n \in \overline{l}_n \right\} \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(f\left(f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^L\right)\right)^{w_i} \right)\right) \cdot f\left(f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\xi}_i^L\right)\right)^{w_i} \right)\right)\right), \\ f^{-1}\left(f\left(f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^U\right)\right)^{w_i} \right)\right) \cdot f\left(f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\xi}_i^U\right)\right)^{w_i} \right)\right)\right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \cdots, \overline{\gamma}_n \in \overline{h}_n, \overline{\xi}_1 \in \overline{l}_1, \cdots, \overline{\xi}_n \in \overline{l} \right\} \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^L\right)\right)^{w_i} \cdot \prod_{i=1}^{n} \left(f\left(\overline{\xi}_i^L\right)\right)^{w_i} \right), \\ f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^U\right)\right)^{w_i} \cdot \prod_{i=1}^{n} \left(f\left(\overline{\xi}_i^U\right)\right)^{w_i} \right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \cdots, \overline{\gamma}_n \in \overline{h}_n, \overline{\xi}_1 \in \overline{l}_1, \cdots, \overline{\xi}_n \in \overline{l} \right\} \\
&= \left\{ \left[\begin{array}{l} f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^L\right) \cdot f\left(\overline{\xi}_i^L\right)\right)^{w_i} \right), \\ f^{-1}\left(\prod_{i=1}^{n} \left(f\left(\overline{\gamma}_i^U\right) \cdot f\left(\overline{\xi}_i^U\right)\right)^{w_i} \right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \cdots, \overline{\gamma}_n \in \overline{h}_n, \overline{\xi}_1 \in \overline{l}_1, \cdots, \overline{\xi}_n \in \overline{l}_n \right\}
\end{aligned}
$$

which completes the proof of Theorem 4.6.

If the multiplicative generator g is assigned different forms, then some specific aggregation operators can be obtained as follows:

Case 1. If $g(t) = \frac{1+t}{t}$, then the GIVHMWA operator reduces to the following form:

$$\text{GIVHMWA}_{\lambda}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)$$
$$=\left\{\left[\frac{\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^{L}\right)^{\lambda w_i}-\prod_{i=1}^{n}\left(\left(1+\bar{\gamma}_i^{L}\right)^{\lambda}-\left(\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}\right)^{1/\lambda}}{\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^{L}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^{L}\right)^{\lambda w_i}-\prod_{i=1}^{n}\left(\left(1+\bar{\gamma}_i^{L}\right)^{\lambda}-\left(\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}\right)^{1/\lambda}},\frac{\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^{U}\right)^{\lambda w_i}-\prod_{i=1}^{n}\left(\left(1+\bar{\gamma}_i^{U}\right)^{\lambda}-\left(\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}\right)^{1/\lambda}}{\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^{U}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^{U}\right)^{\lambda w_i}-\prod_{i=1}^{n}\left(\left(1+\bar{\gamma}_i^{U}\right)^{\lambda}-\left(\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}\right)^{1/\lambda}}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (25)$$

Furthermore, if $\lambda=1$, then the Eq. (25) is transformed to

$$\text{IVHMWA}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left\{\left[\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^{L}\right)^{w_i}-1,\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^{U}\right)^{w_i}-1\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (26)$$

Case 2. If $g(t)=\dfrac{2+t}{t}$, then the GIVHMWA operator reduces to the following form:

$$\text{GIVHMWA}_{\lambda}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)$$
$$=\left\{\left[\frac{2\left(\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{L}\right)^{\lambda}+3\left(\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{L}\right)^{\lambda}-\left(\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}}{\begin{array}{l}\left(4\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{L}\right)^{\lambda}-\left(\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}+\left(\prod_{i=1}^{n}\left(3\left(\bar{\gamma}_i^{L}\right)^{\lambda}+\left(2+\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{L}\right)^{\lambda}-\left(\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}\\-\left(\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{L}\right)^{\lambda}+3\left(\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{L}\right)^{\lambda}-\left(\bar{\gamma}_i^{L}\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}\end{array}},\frac{2\left(\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{U}\right)^{\lambda}+3\left(\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{U}\right)^{\lambda}-\left(\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}}{\begin{array}{l}\left(4\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{U}\right)^{\lambda}-\left(\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}+\left(\prod_{i=1}^{n}\left(3\left(\bar{\gamma}_i^{U}\right)^{\lambda}+\left(2+\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{U}\right)^{\lambda}-\left(\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}\\-\left(\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{U}\right)^{\lambda}+3\left(\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(2+\bar{\gamma}_i^{U}\right)^{\lambda}-\left(\bar{\gamma}_i^{U}\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}\end{array}}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (27)$$

Furthermore, if $\lambda=1$, then the Eq. (27) is transformed to

$$\text{IVHMWA}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left\{\left[\frac{\left(\prod_{i=1}^{n}\left(1+2\bar{\gamma}_i^{L}\right)^{w_i}\right)-1}{2},\frac{\left(\prod_{i=1}^{n}\left(1+2\bar{\gamma}_i^{U}\right)^{w_i}\right)-1}{2}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (28)$$

Case 3. If $g(t)=\dfrac{\theta+t}{t}$, $\theta>0$, then the GIVHMWA operator reduces to the following form:

$$\text{GIVHMWA}_{\lambda}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)$$
$$=\left\{\left[\frac{\theta\left(\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^L\right)^{\lambda}+\left(\theta^2-1\right)\left(\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^L\right)^{\lambda}-\left(\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}}{\left(\theta^2\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^L\right)^{\lambda}-\left(\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}+\left(\prod_{i=1}^{n}\left(\left(\theta^2-1\right)\left(\bar{\gamma}_i^L\right)^{\lambda}+\left(\theta+\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^L\right)^{\lambda}-\left(\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}-\left(\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^L\right)^{\lambda}+\left(\theta^2-1\right)\left(\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^L\right)^{\lambda}-\left(\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}},\right.\right.$$
$$\left.\left.\frac{\theta\left(\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^U\right)^{\lambda}+\left(\theta^2-1\right)\left(\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^U\right)^{\lambda}-\left(\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}}{\left(\theta^2\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^U\right)^{\lambda}-\left(\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}+\left(\prod_{i=1}^{n}\left(\left(\theta^2-1\right)\left(\bar{\gamma}_i^U\right)^{\lambda}+\left(\theta+\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^U\right)^{\lambda}-\left(\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}-\left(\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^U\right)^{\lambda}+\left(\theta^2-1\right)\left(\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(\theta+\bar{\gamma}_i^U\right)^{\lambda}-\left(\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}\right)\right)^{1/\lambda}}\right]\middle|\,\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (29)$$

Furthermore, if $\lambda=1$, then the Eq. (29) is transformed to

$$\text{IVHMWA}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left\{\left[\frac{\left(\prod_{i=1}^{n}\left(1+\theta\bar{\gamma}_i^L\right)^{w_i}\right)-1}{\theta},\frac{\left(\prod_{i=1}^{n}\left(1+\theta\bar{\gamma}_i^U\right)^{w_i}\right)-1}{\theta}\right]\middle|\,\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (30)$$

Especially, if $\theta=1$, then the Eqs. (29) and (30) reduce to Eqs. (25) and (26), respectively; if $\theta=2$, then Eqs. (29) and (30) reduce to Eqs. (27) and (28), respectively.

Example 4.1. Assume that $\bar{h}_1=\{[1/4,1/3],[1/2,1],[1/5,1/3]\}$, $\bar{h}_2=\{[6,8],[5,6]\}$ and $\bar{h}_3=\{[4,5],[6,7]\}$ are three IVHMEs, and their weight vector is $w=(0.2,0.5,0.3)^T$. Let $g(t)=\dfrac{1+t}{t}$, then from Eq. (25), we have

$$\text{GIVHMWA}_1\left(\bar{h}_1,\bar{h}_2,\bar{h}_3\right)=\text{IVHMWA}\left(\bar{h}_1,\bar{h}_2,\bar{h}_3\right)$$
$$=\left\{\begin{array}{l}[3.4836,4.4394],[3.9598,4.9297],[3.1510,3.7971],[3.5918,4.2295],[3.6501,4.8989],\\ [4.1439,5.4306],[3.3051,4.2024],[3.7624,4.6713],[3.4471,4.4394],[3.9194,4.9297],\\ [3.1172,3.7971],[3.5545,4.2295]\end{array}\right\}$$

$$\text{GIVHMWA}_6\left(\bar{h}_1,\bar{h}_2,\bar{h}_3\right)$$
$$=\left\{\begin{array}{l}[4.3765,5.5510],[4.9633,6.1446],[3.9168,4.6771],[4.4885,5.2351],[4.3790,5.5830],\\ [4.9659,6.1781],[3.9192,4.7073],[4.4910,5.2664],[4.3764,5.5510],[4.9632,6.1446],\\ [3.9167,4.6771],[4.4884,5.2351]\end{array}\right\}$$

Example 4.2. If we use the corresponding IVHMEs of the IVHFEs in Example 3.1 to express the input arguments, then we have $\bar{h}_1=\{[9,9]\}$, $\bar{h}_2=\{[1/9,1/9]\}$, $\bar{h}_3=\{[1/9,1/9]\}$,

$\bar{h}_4 = \{[1/9, 1/9]\}$, $\bar{h}_5 = \{[1/9, 1/9]\}$ and $\bar{h}_6 = \{[1/9, 1/9]\}$. Therefore, by the IVHMWA operator (Eq. (18)), we can get

$$\text{IVHMWA}\left(\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4, \bar{h}_5, \bar{h}_6\right) = \{[0.6025, 0.6025]\}$$

which is consistent with our intuition. Therefore, the IVHMEs can contain more original information than the IVHFEs in the aggregation process, which leads to the result that the IVHMEs are more reasonable and reliable than the IVHFEs in dealing with such a situation.

4.2 The GIVHMWG and IVHMWG Operators

Based on the GIVHMWA operator and the geometric mean, we next define a generalized interval-valued hesitant multiplicative weighted geometric (GIVHMWG) operator as:

Definition 4.2. Let $\bar{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and let $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\bar{h}_i$ ($i = 1, 2, \cdots, n$) with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, a generalized interval-valued hesitant multiplicative weighted geometric (GIVHMWG) operator is a mapping $\bar{H}^n \to \bar{H}$, where

$$\text{GIVHMWG}_\lambda\left(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_n\right) = \frac{1}{\lambda}\left(\overset{n}{\underset{i=1}{\otimes}}\left(\lambda \bar{h}_i\right)^{w_i}\right) \tag{31}$$

with $\lambda > 0$.

Especially, if $\lambda = 1$, then the GIVHMWG operator reduces to the interval-valued hesitant multiplicative weighted geometric (IVHMWG) operator:

$$\text{IVHMWG}\left(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_n\right) = \overset{n}{\underset{i=1}{\otimes}} \bar{h}_i^{w_i} \tag{32}$$

Theorem 4.7. Let $\bar{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\bar{h}_i$ ($i = 1, 2, \cdots, n$), where w_i indicates the importance degree of $\bar{h}_i$, satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by using the GIVHMWG operator is also an IVHME, and

$$\text{GIVHMWG}_\lambda\left(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_n\right) = \left\{ \left[\begin{array}{l} f^{-1}\left(\left(f\left(g^{-1}\left(\prod_{i=1}^{n}\left(g\left(f^{-1}\left(\left(f\left(\bar{\gamma}_i^L\right)\right)^{\lambda}\right)\right)\right)^{w_i}\right)\right)\right)^{\frac{1}{\lambda}}\right), \\ f^{-1}\left(\left(f\left(g^{-1}\left(\prod_{i=1}^{n}\left(g\left(f^{-1}\left(\left(f\left(\bar{\gamma}_i^U\right)\right)^{\lambda}\right)\right)\right)^{w_i}\right)\right)\right)^{\frac{1}{\lambda}}\right) \end{array} \right] \middle| \bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n \right\} \tag{33}$$

Based on Theorem 4.7, when $\lambda = 1$, the following theorem can be easily obtained:

Theorem 4.8. Let $\overline{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\overline{h}_i$ ($i = 1, 2, \cdots, n$), where w_i indicates the importance degree of $\overline{h}_i$, satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by using the IVHMWG operator is also an IVHME, and

$$\text{IVHMWG}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right) = \left\{ \left[g^{-1}\left(\prod_{i=1}^{n} \left(g\left(\overline{\gamma}_i^L\right)\right)^{w_i} \right), g^{-1}\left(\prod_{i=1}^{n} \left(g\left(\overline{\gamma}_i^U\right)\right)^{w_i} \right) \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\} \quad (34)$$

Especially, if $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$, then the GIVHMWG operator reduces to the generalized interval-valued hesitant multiplicative geometric (GIVHMG) operator:

$$\text{GIVHMG}_\lambda\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right) = \frac{1}{\lambda}\left(\bigotimes_{i=1}^{n} \left(\lambda \overline{h}_i\right)^{\frac{1}{n}} \right)$$
$$= \left\{ \left[\begin{array}{l} f^{-1}\left(\left(f\left(g^{-1}\left(\prod_{i=1}^{n} \left(g\left(f^{-1}\left(\left(f\left(\overline{\gamma}_i^L\right)\right)^{\lambda} \right) \right) \right)^{\frac{1}{n}} \right) \right) \right)^{\frac{1}{\lambda}} \right), \\ f^{-1}\left(\left(f\left(g^{-1}\left(\prod_{i=1}^{n} \left(g\left(f^{-1}\left(\left(f\left(\overline{\gamma}_i^U\right)\right)^{\lambda} \right) \right) \right)^{\frac{1}{n}} \right) \right) \right)^{\frac{1}{\lambda}} \right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\} \quad (35)$$

If $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$, then the IVHMWG operator reduces to the interval-valued hesitant multiplicative geometric (IVHMG) operator:

$$\text{IVHMG}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right) = \bigotimes_{i=1}^{n}\left(\lambda \overline{h}_i^{\frac{1}{n}} \right)$$
$$= \left\{ \left[g^{-1}\left(\prod_{i=1}^{n} \left(g\left(\overline{\gamma}_i^L\right)\right)^{\frac{1}{n}} \right), g^{-1}\left(\prod_{i=1}^{n} \left(g\left(\overline{\gamma}_i^U\right)\right)^{\frac{1}{n}} \right) \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\} \quad (36)$$

In what follows, we investigate some desirable properties of the IVHMWG operator.

Theorem 4.9. Let $\overline{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\overline{h}_i$ ($i = 1, 2, \cdots, n$), satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, if $\overline{h}$ is an IVHME, then

$$\text{IVHMWG}\left(\overline{h}_1 \otimes \overline{h}, \overline{h}_2 \otimes \overline{h}, \cdots, \overline{h}_n \otimes \overline{h}\right) = \text{IVHMWG}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right) \otimes \overline{h} \quad (37)$$

Theorem 4.10. Let $\overline{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\overline{h}_i$ ($i = 1, 2, \cdots, n$), satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, if $r > 0$, then

$$\text{IVHMWG}\left(\overline{h}_1^r, \overline{h}_2^r, \cdots, \overline{h}_n^r\right) = \left(\text{IVHMWG}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right)\right)^r \tag{38}$$

According to Theorems 4.9 and 4.10, we can easily obtain Theorem 4.11:

Theorem 4.11. Let $\overline{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\overline{h}_i$ ($i = 1, 2, \cdots, n$), satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, if $r > 0$ and $\overline{h}$ is an IVHME, then

$$\text{IVHMWG}\left(\overline{h}_1^r \otimes \overline{h}, \overline{h}_2^r \otimes \overline{h}, \cdots, \overline{h}_n^r \otimes \overline{h}\right) = \left(\text{IVHMWG}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right)\right)^r \otimes \overline{h} \tag{39}$$

Theorem 4.12. Let $\overline{h}_i$ and $\overline{l}_i$ ($i = 1, 2, \cdots, n$) be two collections of IVHMEs, $w = (w_1, w_2, \cdots, w_n)^T$ be their weight vector with $w_i \in [0,1]$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^{n} w_i = 1$, then

$$\text{IVHMWG}\left(\overline{h}_1 \otimes \overline{l}_1, \overline{h}_2 \otimes \overline{l}_2, \cdots, \overline{h}_n \otimes \overline{l}_n\right) = \text{IVHMWG}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right) \otimes \text{IVHMWG}\left(\overline{l}_1, \overline{l}_2, \cdots, \overline{l}_n\right) \tag{40}$$

In what follows, we will investigate the relationship between GIVHMWA operator and GIVHMWG operator.

Theorem 4.13. Let $\overline{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\overline{h}_i$ ($i = 1, 2, \cdots, n$), satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then we have

(1) $$\text{GIVHMWA}_\lambda\left(\overline{h}_1^c, \overline{h}_2^c, \cdots, \overline{h}_n^c\right) = \left(\text{GIVHMWG}_\lambda\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right)\right)^c \tag{41}$$

(2) $$\text{GIVHMWG}_\lambda\left(\overline{h}_1^c, \overline{h}_2^c, \cdots, \overline{h}_n^c\right) = \left(\text{GIVHMWA}_\lambda\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right)\right)^c \tag{42}$$

Proof. (1) According to Eqs. (7), (16), and (33), we can get

$$\text{GIVHMWA}_\lambda\left(\overline{h}_1^c, \overline{h}_2^c, \cdots, \overline{h}_n^c\right)$$

$$= \left\{ \left[\begin{array}{l} g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n} \left(f\left(g^{-1}\left(\left(g\left(1/\overline{\gamma}_i^U \right) \right)^\lambda \right) \right) \right)^{w_i} \right) \right) \right)^{1/\lambda} \right), \\ g^{-1}\left(\left(g\left(f^{-1}\left(\prod_{i=1}^{n} \left(f\left(g^{-1}\left(\left(g\left(1/\overline{\gamma}_i^L \right) \right)^\lambda \right) \right) \right)^{w_i} \right) \right) \right)^{1/\lambda} \right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\}$$

$$= \left\{ \left[\begin{array}{l} \dfrac{1}{f^{-1}\left(\left(f\left(g^{-1}\left(\prod_{i=1}^{n} \left(g\left(f^{-1}\left(\left(f\left(\overline{\gamma}_i^U \right) \right)^\lambda \right) \right) \right)^{w_i} \right) \right) \right)^{1/\lambda} \right)}, \\ \dfrac{1}{f^{-1}\left(\left(f\left(g^{-1}\left(\prod_{i=1}^{n} \left(g\left(f^{-1}\left(\left(f\left(\overline{\gamma}_i^L \right) \right)^\lambda \right) \right) \right)^{w_i} \right) \right) \right)^{1/\lambda} \right)} \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\}$$

$$= \left\{ \left[\begin{array}{l} f^{-1}\left(\left(f\left(g^{-1}\left(\prod_{i=1}^{n} \left(g\left(f^{-1}\left(\left(f\left(\overline{\gamma}_i^L \right) \right)^\lambda \right) \right) \right)^{w_i} \right) \right) \right)^{\frac{1}{\lambda}} \right), \\ f^{-1}\left(\left(f\left(g^{-1}\left(\prod_{i=1}^{n} \left(g\left(f^{-1}\left(\left(f\left(\overline{\gamma}_i^U \right) \right)^\lambda \right) \right) \right)^{w_i} \right) \right) \right)^{\frac{1}{\lambda}} \right) \end{array} \right] \middle| \overline{\gamma}_1 \in \overline{h}_1, \overline{\gamma}_2 \in \overline{h}_2, \cdots, \overline{\gamma}_n \in \overline{h}_n \right\}^c$$

$$= \left(\text{GIVHMWG}_\lambda\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right)\right)^c$$

Similarly, we can prove Eq. (42), which completes the proof of Theorem 4.13.

Theorem 4.14. Let $\overline{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHMEs, and $w = \left(w_1, w_2, \cdots, w_n\right)^T$ be the weight vector of $\overline{h}_i$ ($i = 1, 2, \cdots, n$), satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then we have

$$\text{(1) IVHMWA}\left(\overline{h}_1^c, \overline{h}_2^c, \cdots, \overline{h}_n^c\right) = \left(\text{IVHMWG}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right)\right)^c \tag{43}$$

$$\text{(2) IVHMWG}\left(\overline{h}_1^c, \overline{h}_2^c, \cdots, \overline{h}_n^c\right) = \left(\text{IVHMWA}\left(\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n\right)\right)^c \tag{44}$$

If the multiplicative generator g is assigned different forms, then some specific aggregation operators can be obtained as follows:

Case 1. If $g(t) = \dfrac{1+t}{t}$, then the GIVHMWG operator reduces to the following form:

$$\mathrm{GIVHMWG}_{\lambda}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)$$
$$=\left\{\left[\frac{\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^L\right)^{\lambda w_i}\right)^{\frac{1}{\lambda}}-\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^L\right)^{\lambda w_i}-\prod_{i=1}^{n}\left(\left(1+\bar{\gamma}_i^L\right)^{\lambda}-1\right)^{w_i}\right)^{\frac{1}{\lambda}}}{\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^L\right)^{\lambda w_i}-\prod_{i=1}^{n}\left(\left(1+\bar{\gamma}_i^L\right)^{\lambda}-1\right)^{w_i}\right)^{\frac{1}{\lambda}}},\frac{\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^U\right)^{\lambda w_i}\right)^{\frac{1}{\lambda}}-\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^U\right)^{\lambda w_i}-\prod_{i=1}^{n}\left(\left(1+\bar{\gamma}_i^U\right)^{\lambda}-1\right)^{w_i}\right)^{\frac{1}{\lambda}}}{\left(\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^U\right)^{\lambda w_i}-\prod_{i=1}^{n}\left(\left(1+\bar{\gamma}_i^U\right)^{\lambda}-1\right)^{w_i}\right)^{\frac{1}{\lambda}}}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (45)$$

Furthermore, if $\lambda=1$, then the Eq. (45) is transformed to

$$\mathrm{IVHMWG}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left\{\left[\frac{\prod_{i=1}^{n}\left(\bar{\gamma}_i^L\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^L\right)^{w_i}-\prod_{i=1}^{n}\left(\bar{\gamma}_i^L\right)^{w_i}},\frac{\prod_{i=1}^{n}\left(\bar{\gamma}_i^U\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\bar{\gamma}_i^U\right)^{w_i}-\prod_{i=1}^{n}\left(\bar{\gamma}_i^U\right)^{w_i}}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (46)$$

Case 2. If $g(t)=\frac{2+t}{t}$, then the GIVHMWA operator reduces to the following form:

$$\mathrm{GIVHMWG}_{\lambda}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)$$
$$=\left\{\left[\frac{\left(3\prod_{i=1}^{n}\left(\left(1+2\bar{\gamma}_i^L\right)^{\lambda}-1\right)^{w_i}+\prod_{i=1}^{n}\left(3+\left(1+2\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}-\left(\prod_{i=1}^{n}\left(3+\left(1+2\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}-\prod_{i=1}^{n}\left(\left(1+2\bar{\gamma}_i^L\right)^{\lambda}-1\right)^{w_i}\right)^{\frac{1}{\lambda}}}{2\left(\prod_{i=1}^{n}\left(3+\left(1+\theta\bar{\gamma}_i^L\right)^{\lambda}\right)^{w_i}-\prod_{i=1}^{n}\left(\left(1+2\bar{\gamma}_i^L\right)^{\lambda}-1\right)^{w_i}\right)^{\frac{1}{\lambda}}},\frac{\left(3\prod_{i=1}^{n}\left(\left(1+2\bar{\gamma}_i^U\right)^{\lambda}-1\right)^{w_i}+\prod_{i=1}^{n}\left(3+\left(1+2\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}\right)^{\frac{1}{\lambda}}-\left(\prod_{i=1}^{n}\left(3+\left(1+2\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}-\prod_{i=1}^{n}\left(\left(1+2\bar{\gamma}_i^U\right)^{\lambda}-1\right)^{w_i}\right)^{\frac{1}{\lambda}}}{2\left(\prod_{i=1}^{n}\left(3+\left(1+\theta\bar{\gamma}_i^U\right)^{\lambda}\right)^{w_i}-\prod_{i=1}^{n}\left(\left(1+2\bar{\gamma}_i^U\right)^{\lambda}-1\right)^{w_i}\right)^{\frac{1}{\lambda}}}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \quad (47)$$

Furthermore, if $\lambda=1$, then the Eq. (47) is transformed to

$$\text{IVHMWG}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left\{\left[\frac{2\prod_{i=1}^{n}\left(\bar{\gamma}_i^{L}\right)^{w_i}}{\prod_{i=1}^{n}\left(2+\bar{\gamma}_i^{L}\right)^{w_i}-\prod_{i=1}^{n}\left(\bar{\gamma}_i^{L}\right)^{w_i}},\frac{2\prod_{i=1}^{n}\left(\bar{\gamma}_i^{U}\right)^{w_i}}{\prod_{i=1}^{n}\left(2+\bar{\gamma}_i^{U}\right)^{w_i}-\prod_{i=1}^{n}\left(\bar{\gamma}_i^{U}\right)^{w_i}}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \tag{48}$$

Case 3. If $g\left(t\right)=\dfrac{\theta+t}{t}$, $\theta>0$, then the GIVHMWG operator reduces to the following form:

$$\text{GIVHMWG}_{\lambda}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)$$

$$=\left\{\left[\frac{\left(\left(\theta^2-1\right)\left(\prod_{i=1}^{n}\left(\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)+\left(\prod_{i=1}^{n}\left(\theta^2+\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)\right)^{\frac{1}{\lambda}}-\left(\left(\prod_{i=1}^{n}\left(\theta^2+\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)\right)^{\frac{1}{\lambda}}}{\theta\left(\left(\prod_{i=1}^{n}\left(\theta^2+\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)\right)^{\frac{1}{\lambda}}},\frac{\left(\left(\theta^2-1\right)\left(\prod_{i=1}^{n}\left(\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)+\left(\prod_{i=1}^{n}\left(\theta^2+\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)\right)^{\frac{1}{\lambda}}-\left(\left(\prod_{i=1}^{n}\left(\theta^2+\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)\right)^{\frac{1}{\lambda}}}{\theta\left(\left(\prod_{i=1}^{n}\left(\theta^2+\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)-\left(\prod_{i=1}^{n}\left(\left(1+\theta\bar{\gamma}_i^{L}\right)^{\lambda}-1\right)^{w_i}\right)\right)^{\frac{1}{\lambda}}}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \tag{49}$$

Furthermore, if $\lambda=1$, then the Eq. (49) is transformed to

$$\text{IVHMWG}\left(\bar{h}_1,\bar{h}_2,\cdots,\bar{h}_n\right)=\left\{\left[\frac{\theta\prod_{i=1}^{n}\left(\bar{\gamma}_i^{L}\right)^{w_i}}{\prod_{i=1}^{n}\left(\theta+\bar{\gamma}_i^{L}\right)^{w_i}-\prod_{i=1}^{n}\left(\bar{\gamma}_i^{L}\right)^{w_i}},\frac{\theta\prod_{i=1}^{n}\left(\bar{\gamma}_i^{U}\right)^{w_i}}{\prod_{i=1}^{n}\left(\theta+\bar{\gamma}_i^{U}\right)^{w_i}-\prod_{i=1}^{n}\left(\bar{\gamma}_i^{U}\right)^{w_i}}\right]\middle|\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n\right\} \tag{50}$$

Especially, if $\theta=1$, then the Eqs. (49) and (50) reduce to Eqs. (45) and (46), respectively; if $\theta=2$, then Eqs. (49) and (50) reduce to Eqs. (47) and (48), respectively.

Example 4.3. Assume that $\overline{h}_1=\{[1/8,1/6],[1/7,1/5],[1/4,1/3]\}$, $\overline{h}_2=\{[7,9],[5,6]\}$ and $\overline{h}_3=\{[1/5,1/3],[1/2,1]\}$ are three IVHMEs, and their weight vector is $w=(0.6,0.3,0.1)^T$. Let $g(t)=\frac{1+t}{t}$, then from Eq. (33), we have

$$\text{GIVHMWG}_1\left(\overline{h}_1,\overline{h}_2,\overline{h}_3\right)=\text{IVHMWG}\left(\overline{h}_1,\overline{h}_2,\overline{h}_3\right)$$
$$=\left\{\begin{matrix}[0.2737,0.3558],[0.2993,0.3913],[0.2687,0.3488],[0.2936,0.3835],[0.2998,0.4042],\\ [0.3284,0.4462],[0.2941,0.3960],[0.3221,0.4369],[0.4405,0.5801],[0.4875,0.6488],\\ [0.4313,0.5669],[0.4770,0.6334]\end{matrix}\right\}$$

$$\text{GIVHMWG}_{0.5}\left(\overline{h}_1,\overline{h}_2,\overline{h}_3\right)$$
$$=\left\{\begin{matrix}[0.3016,0.3947],[0.3304,0.4367],[0.2920,0.3806],[0.3198,0.4208],[0.3298,0.4473],\\ [0.3618,0.4963],[0.3192,0.4310],[0.3500,0.4778],[0.4813,0.6372],[0.5323,0.7143],\\ [0.4646,0.6119],[0.5134,0.6850]\end{matrix}\right\}$$

5. Interval-valued Hesitant Multiplicative Preference Relations (IVHMPRS)

In the process of group decision making under uncertainty and vagueness, the decision makers usually compare each pair of alternatives, and provide their preferences by interval multiplicative values, and thus construct the interval multiplicative preference relations (IMPRs) as follows:

Definition 5.1 [32]. Let $X=\{x_1,x_2,\cdots,x_n\}$ be a discrete set of alternatives. An interval multiplicative preference relation (IMPR) R on the set X is defined as $R=\left(r_{ij}\right)_{n\times n}\subset X\times X$, where $r_{ij}=\left[r_{ij}^L,r_{ij}^U\right]$ shows the interval-valued preference degree of the alternative x_i over x_j , and satisfies

$$\frac{1}{9}\le r_{ij}^L\le r_{ij}^U\le 9,\quad r_{ij}^L\cdot r_{ji}^U=r_{ij}^U\cdot r_{ji}^L=1,\quad r_{ii}^L=r_{ii}^U=1,\qquad \forall i,j=1,2,\cdots,n. \tag{51}$$

Generally speaking, a common approach for MAGDM with IMPRs first aggregates the individual IMPRs into the group IMPR, which often causes the loss of information. To overcome this drawback, instead of performing information aggregation, we first collect all of the possible interval multiplicative preference values provided by the decision makers (DMs) into an interval-valued hesitant multiplicative preference relation, which can avoid the loss of information and fully reflects the differences of preference information of different DMs.

If some decision makers provide some interval multiplicative preference values to describe the degrees that x_i is preferred to x_j , which are denoted by $\overline{r}_{ij}^1$, $\overline{r}_{ij}^2$, $\cdots$, $\overline{r}_{ij}^{l_{ij}}$, then the degrees that

x_i is preferred to x_j can be represented by an IVHME $\overline{r}_{ij} = \left\{ \overline{r}_{ij}^1, \overline{r}_{ij}^2, \cdots, \overline{r}_{ij}^{l_{ij}} \right\}$. All $\overline{r}_{ij}$ ($i, j = 1, 2, \cdots, n$) constitute an interval-valued hesitant multiplicative preference relation, which is defined below:

Definition 5.2. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a discrete set of alternatives. An interval-valued hesitant multiplicative preference relation (IVHMPR) on X is denoted by a matrix $\overline{R} = \left(\overline{r}_{ij}\right)_{n \times n} \subset X \times X$, where $\overline{r}_{ij} = \left\{ \overline{r}_{ij}^s \middle| s = 1, 2, \cdots, l_{\overline{r}_{ij}} \right\}$ is an IVHME, indicating all possible degrees to which x_i is preferred to x_j, and $l_{\overline{r}_{ij}}$ represents the number of intervals in $\overline{r}_{ij}$. Moreover, $\overline{r}_{ij}$ should satisfy

$$\inf \overline{r}_{ij}^{\sigma(s)} \times \sup \overline{r}_{ji}^{\sigma\left(l_{\overline{r}_{ij}} - s + 1\right)} = \sup \overline{r}_{ij}^{\sigma(s)} \times \inf \overline{r}_{ji}^{\sigma\left(l_{\overline{r}_{ij}} - s + 1\right)} = 1, \ \overline{r}_{ii} = \left\{ [1,1] \right\}, \forall i, j = 1, 2, \cdots, n \tag{52}$$

where the elements in $\overline{r}_{ij}$ are arranged in an increasing order, $\overline{r}_{ij}^{\sigma(s)}$ denotes the *s*th smallest value in $\overline{r}_{ij}$, and $\inf \overline{r}_{ij}^{\sigma(s)}$ and $\sup \overline{r}_{ij}^{\sigma(s)}$ denote the lower and upper limits of $\overline{r}_{ij}^{\sigma(s)}$, respectively.

For a multi-criteria decision making (MCDM) problem based on interval-valued hesitant multiplicative preference relations (IVHMPRs), let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of n alternatives, $C = \{c_1, c_2, \ldots, c_m\}$ be a set of m criteria, whose weight vector is $w = (w_1, w_2, \cdots, w_m)^T$ satisfying $w_k \in [0,1]$, $k = 1, 2, \cdots, m$, and $\sum_{k=1}^{m} w_k = 1$, where w_k denotes the importance degree of the criterion c_k. The decision makers provide all the possible interval multiplicative preference values to which x_i is preferred to x_j with respect to the criterion c_k represented by the IVHME $\overline{r}_{ij}^{(k)}$. All $\overline{r}_{ij}^{(k)}$ ($i, j = 1, 2, \cdots, n$) construct the IVHMPR $\overline{R}^{(k)} = \left(\overline{r}_{ij}^{(k)}\right)_{n \times n}$ with respect to the criterion c_k. To get the best alternative, the following steps are involved:

Algorithm 5.1.

Step 1. Utilize the GIVHMA (or GIVHMG) operator to aggregate all $\overline{r}_{ij}^{(k)}$ ($j = 1, 2, \cdots, n$) that correspond to the alternative x_i, and get the IVHME $\overline{r}_i^{(k)}$ of the alternative x_i over all the other alternatives for the criterion c_k.

Step 2. Utilize the GIVHMWA (or the GIVHMWG) operator to aggregate all $\overline{r}_i^{(k)}$ ($k = 1, 2, \cdots, m$) into an overall IVHME $\overline{r}_i$ for the alternative x_i.

Step 3. Compute the score functions $s\left(\overline{r}_i\right)$ of $\overline{r}_i$ ($i=1,2,\cdots,n$) by Definition 3.3, and rank all the alternatives x_i ($i=1,2,\cdots,n$) according to $s\left(\overline{r}_i\right)$ in descending order.
Step 4. End.

6. Illustrative Example

6.1. An Illustrative Example

In this subsection, a practical example adapted from [22,23] is employed to demonstrate the validity of the developed approach.

Example 6.1 [22,23]. Let us consider a factory which intends to select a new site for new buildings. Four alternatives x_i ($i=1,2,3,4$) are available, and three decision makers compare these four alternatives with respect to the three criteria: (1) c_1 (price); (2) c_2 (location); and (3) c_3 (environment). The weight vector of three criteria c_k ($k=1,2,3$) is $w=\left(0.2,0.5,0.3\right)^T$. The selection of the new site can be modeled as a hierarchical structure, as shown in Fig. 1. The three decision makers provide all the possible interval multiplicative preference values to which x_i is preferred to x_j with respect to the criterion c_k represented by the IVHME $\overline{r}_{ij}^{(k)}$. All $\overline{r}_{ij}^{(k)}$ ($i,j=1,2,3,4$) are contained in the IVHMPR $\overline{R}^{(k)}=\left(\overline{r}_{ij}^{(k)}\right)_{n\times n}$ with respect to the criterion c_k (see Tables 2-4). In the following, we explain how the IVHMPR $\overline{R}^{(k)}=\left(\overline{r}_{ij}^{(k)}\right)_{n\times n}$ is obtained. Take $\overline{R}^{(1)}=\left(\overline{r}_{ij}^{(1)}\right)_{n\times n}$ as an example. The three decision makers provide their preference information that x_1 is preferred to x_4 with respect to the criterion c_1 in the form of interval multiplicative values. Suppose that one decision maker provides $\left[1,3\right]$, one decision maker provides $\left[3,4\right]$, and the third decision maker provides $\left[3,5\right]$. Considering that three decision makers cannot persuaded each other to change their opinions, the preference information that x_1 is preferred to x_4 can be considered as an IVHME, i.e., $\overline{r}_{14}^{(1)}=\left\{\left[1,3\right],\left[3,4\right],\left[3,5\right]\right\}$. Similarly we can denote the symmetric element $\overline{r}_{41}^{(1)}$ of $\overline{r}_{14}^{(1)}$ as $\overline{r}_{41}^{(1)}=\left\{\left[1/5,1/3\right],\left[1/4,1/3\right],\left[1/3,1\right]\right\}$. Other symmetric elements $\overline{r}_{ij}^{(1)}$ and $\overline{r}_{ji}^{(1)}$ in $\overline{R}^{(1)}$ can be obtained in an analogous way. Moreover, $\overline{r}_{ii}^{(1)}$ represents the preference degree to which x_i is preferred to itself with respect to the criterion c_1; that is, it is equally preferred, so $\overline{r}_{ii}^{(1)}=\left\{\left[1,1\right]\right\}$. The IVHMPR $\overline{R}^{(1)}$ is obtained through the above procedure. Similarly, we can get the IVHMPRs $\overline{R}^{(2)}$ and $\overline{R}^{(3)}$.

Table 2. The IVHMPR $\overline{R}^{(1)}$ with respect to the criterion c_1 .

2	x_1	x_2	x_3	x_4
x_1	{[1,1]}	{[3, 4], [4, 5]}	{[1/3, 1/2]}	{[1, 3], [3, 4], [3, 5]}
x_2	{[1/5, 1/4], [1/4, 1/3]}	{[1,1]}	{[3, 4], [3, 5]}	{[1/2, 1]}
x_3	{[2, 3]}	{[1/5, 1/3], [1/4, 1/3]}	{[1,1]}	{[1/3, 1], [1/2, 1]}
x_4	{[1/5, 1/3], [1/4, 1/3], [1/3, 1]}	{[1, 2]}	{[1, 2], [1, 3]}	{[1,1]}

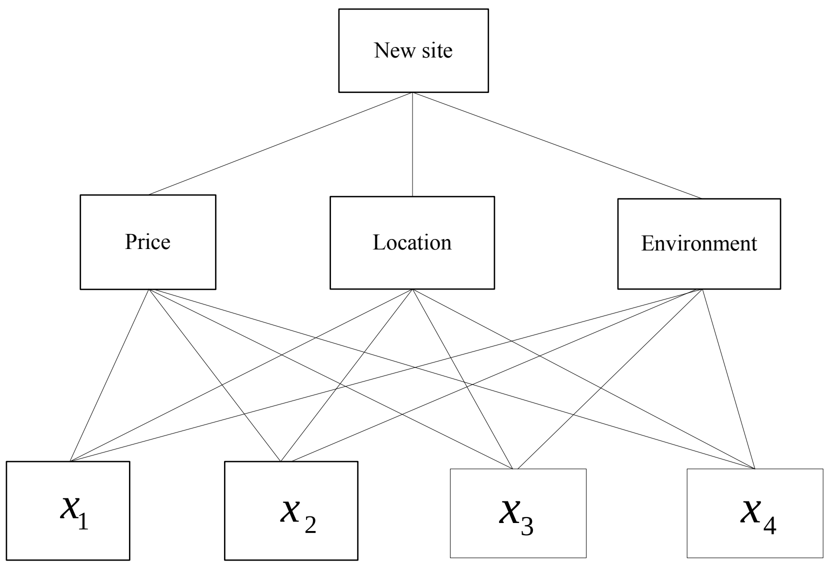

Fig. 1. Hierarchical structure.

Table 3. The IVHMPR $\overline{R}^{(2)}$ with respect to the criterion c_2 .

3	x_1	x_2	x_3	x_4
x_1	{[1,1]}	{[1/2, 1], [2, 3]}	{1/4, 1/3]}	{[3, 4], [3, 5]}
x_2	{[1/3, 1/2], [1, 2]}	{[1,1]}	{[1/5, 1/4], [1/4, 1/3], [1/2, 1]}	{[6, 7]}
x_3	{[3, 4]}	{[1, 2], [3, 4], [4, 5]}	{[1,1]}	{[1/6, 1/5], [1/5, 1/4], [1/5, 1/3]}
x_4	{[1/5, 1/3], [1/4, 1/3]}	{[1/7, 1/6]}	{[3, 5], [4, 5], [5, 6]}	{[1,1]}

Table 4. The IVHMPR $\overline{R}^{(3)}$ with respect to the criterion c_3

4	x_1	x_2	x_3	x_4
x_1	{[1,1]}	{ [1/7, 1/5], [1/6, 1/4]}	{[1/9, 1/8]}	{[1, 3], [2, 3], [3, 4]}
x_2	{[4, 6], [5, 7]}	{[1,1]}	{[5, 7], [6, 7]}	{[1/4, 1/3]}
x_3	{[8, 9]}	{[1/7, 1/6], [1/7, 1/5]}	{[1,1]}	{[3, 5]}
x_4	{[1/4, 1/3], [1/3, 1/2], [1/3, 1]}	{[3, 4]}	{[1/5, 1/3]}	{[1,1]}

Table 5. The aggregation results of the alternatives x_i ($i=1,2,3,4$) with respect to the criteria c_k ($k=1,2,3$).

5	c_1	c_2	c_3
x_1	{[1.1491,1.7832],[1.5558,1.9428],[1.5558,2.0801],[1.2724,1.9130],[1.7024,2.0801],[1.7024,2.2237]}	{[0.9680,1.2724],[0.9680,1.3784],[1.3403,1.7024],[1.3403,1.8284]}	{[0.5012,0.8128],[0.6614,0.8128],[0.7853,0.9168],[0.5090,0.8314],[0.6700,0.8314],[0.7945,0.9365]}
x_2	{[0.9480,1.2361],[0.9480,1.3403],[0.9680,1.2724],[0.9680,1.3784]}	{[1.1755,1.3403],[1.1978,1.3784],[1.3003,1.6321],[1.4076,1.7832],[1.4323,1.8284],[1.5457,2.1302]}	{[1.9428,2.4957],[2.0585,2.4957],[2.0801,2.6144],[2.2011,2.6144]}
x_3	{[0.7602,1.1491],[0.8128,1.1491],[0.7783,1.1491],[0.8314,1.1491]}	{[1.0786,1.4495],[1.0933,1.4746],[1.0933,1.5149],[1.4719,1.7832],[1.4893,1.8117],[1.4893,1.8574],[1.6137,1.9130],[1.6321,1.9428],[1.6321,1.9907]}	{[2.0118,2.4398],[2.0118,2.4641]}
x_4	{[0.7602,1.2134],[0.7602,1.3784],[0.7783,1.2134],[0.7783,1.3784],[0.8072,1.4495],[0.8072,1.6321]}	{[0.8200,1.0786],[0.9244,1.0786],[1.0141,1.1602],[0.8386,1.0786],[0.9441,1.0786],[1.0348,1.1602]}	{[0.8612,1.0534],[0.8915,1.1147],[0.8915,1.2724]}

Let $g(t)=\frac{1+t}{t}$. To obtain the ranking of the alternatives, the following steps are given:

Step 1. Utilize the GIVHMA operator (without loss of generality, let $\lambda=1$) to aggregate all $\overline{r}_{ij}^{(k)}$ ($j=1,2,3,4$), and get the IVHME $\overline{r}_i^{(k)}$ of the alternative x_i with respect to the criterion c_k (see Table 5). For example,

$$
\begin{aligned}
\overline{r}_4^{(3)} &= \text{GIVHMA}_1\left(\overline{r}_{41}^{(3)},\overline{r}_{42}^{(3)},\overline{r}_{43}^{(3)},\overline{r}_{44}^{(3)}\right) \\
&= \text{IVHMA}\left(\overline{r}_{41}^{(3)},\overline{r}_{42}^{(3)},\overline{r}_{43}^{(3)},\overline{r}_{44}^{(3)}\right) \\
&= \text{IVHMA}\left(\{[1/4,1/3],[1/3,1/2],[1/3,1]\},\{[3,4]\},\{[1/5,1/3]\},\{[1,1]\}\right) \\
&= \{[0.8612,1.0534],[0.8915,1.1147],[0.8915,1.2724]\}
\end{aligned}
$$

Step 2. Utilize the GIVHMWA operator to aggregate all $\overline{r}_i^{(k)}$ ($k=1,2,3$) into an overall IVHME $\overline{r}_i$ for the alternative x_i, which are not shown here due to the limited space.

Step 3. By Definition 3.3, we compute the score functions $s(\overline{r}_i)$ of $\overline{r}_i$ ($i=1,2,3,4$) as follows:

$$s(\overline{r}_1)=1.2000,\quad s(\overline{r}_2)=1.6248,\quad s(\overline{r}_3)=1.5952,\quad s(\overline{r}_4)=1.0158$$

Since $s(\overline{r}_2)>s(\overline{r}_3)>s(\overline{r}_1)>s(\overline{r}_4)$, then the ranking of the alternatives is $x_2 > x_3 > x_1 > x_4$, which shows x_2 is the best among four alternatives.

It is noted that the above results are obtained under the assumption that $\lambda=1$. In the following, we will analyze the variation of the ranking of the alternatives with respect to the different values of the parameter λ. Fig. 2 shows the score functions of the alternatives obtained by the GIVHMWA operator, from which we can find that the score functions of each alternatives increase as the values of λ increase from 0 to 20, and

(1) when $\lambda \in (0,10.4163]$, the ranking of the four alternatives is $x_2 > x_3 > x_1 > x_4$ and the best choice is x_2.

(2) when $\lambda \in (10.4163,20]$, the ranking of the four alternatives is $x_2 > x_3 > x_4 > x_1$ and the best choice is x_2.

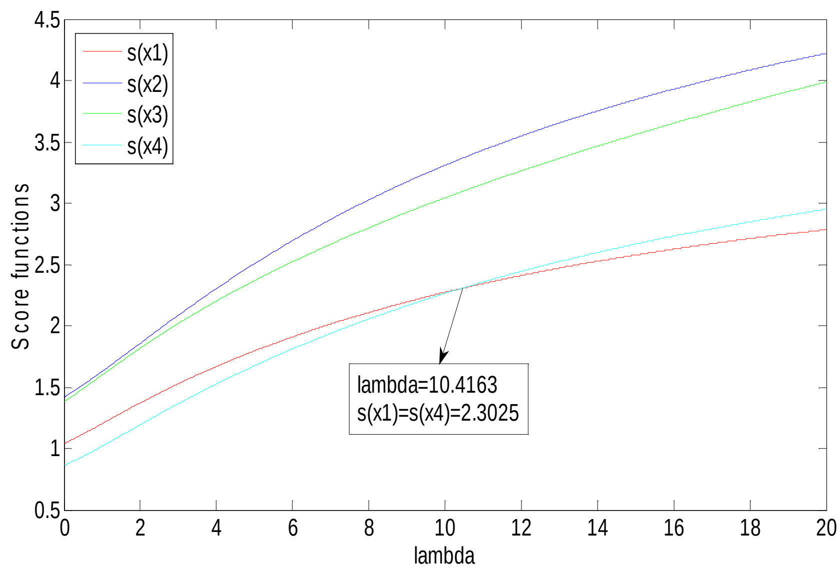

Fig. 2. Score functions for alternatives obtained by the GIVHMWA operator.

If we use the GIVHMWG operator instead of the GIVHMWA operator to aggregate the interval-valued hesitant multiplicative preference information, then the score functions of the alternatives are shown in Fig. 3, from which we can see that the score functions of each alternatives obtained by the GIVHMWG operator decrease as the parameter λ changes from 0 to 20. From Fig. 3, we can also find that

(1) when $\lambda \in (0, 12.5818]$, the ranking of the four alternatives is $x_2 > x_3 > x_1 > x_4$ and the best choice is x_2.
(2) when $\lambda \in (12.5818, 20]$, the ranking of the four alternatives is $x_2 > x_3 > x_4 > x_1$ and the best choice is x_2.

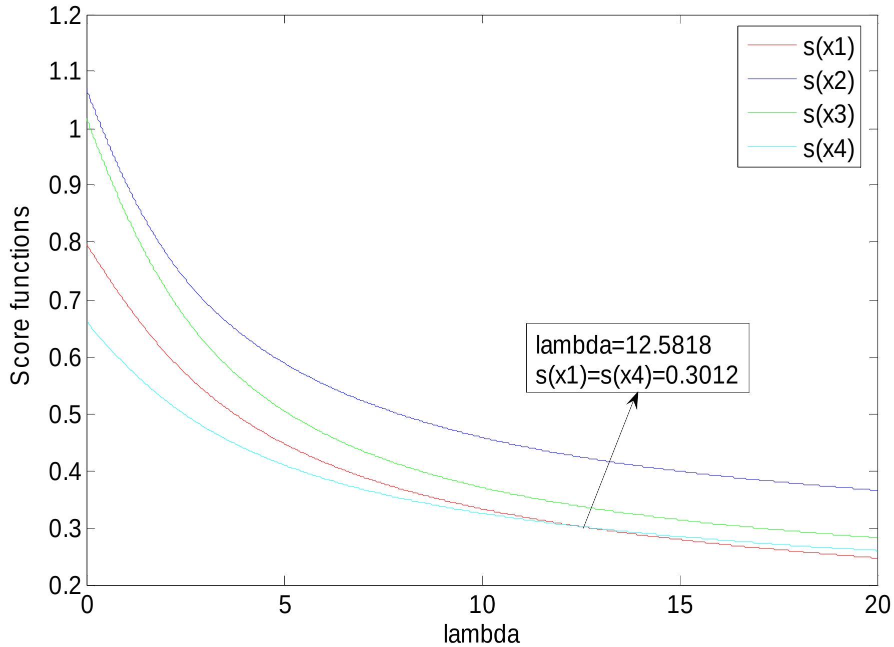

Fig. 3. Score functions for alternatives obtained by the GIVHMWG operator.

Fig. 4 illustrates the deviation values between the score functions obtained by the GIVHMWA operator and the ones obtained by the GIVHMWG operator, from which we can find that the values obtained by the GIVHMWA operator are much bigger than the ones obtained by the GIVHMWG operator for the same value of the parameter λ and the same aggregation values, and the deviation values increase as the value of the parameter λ increases.

Fig. 4 indicates that the GIVHMWA operator can obtain more favorable (or optimistic) expectations, and therefore can be considered as an optimistic operator, while the GIVHMWG operator can obtain more unfavorable (or pessimistic) expectations, and therefore can be considered as a pessimistic operator. The values of the parameter λ can be referred to as the optimistic or pessimistic levels. From Figs. 2, 3 and 4, we can conclude that the decision makers who are optimistic could use the GIVHMWA operator and choose the bigger values of the parameter λ, while the decision makers who are pessimistic could use the GIVHMWG operator and choose the bigger values of the parameter λ.

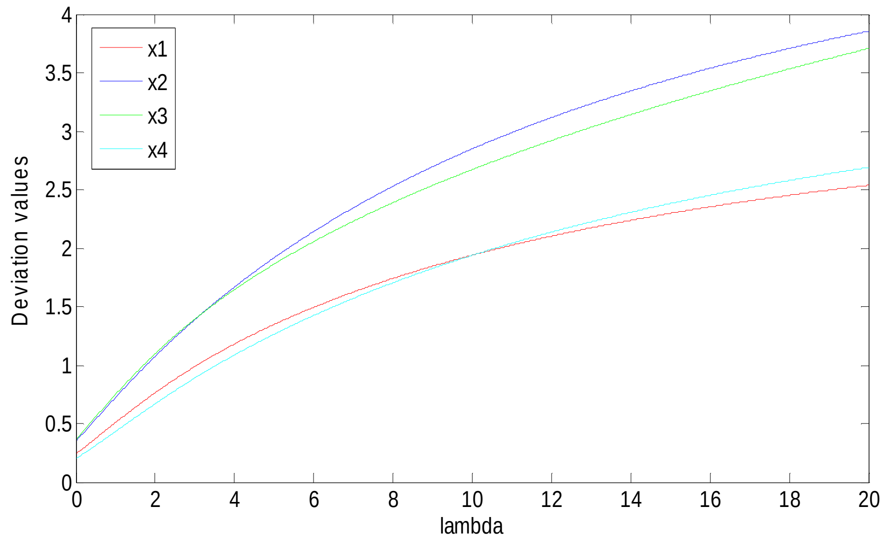

Fig. 4. Deviation values for alternatives between the GIVHMWA and GIVHMWG operators.

6.2. Comparison with the Existing Approach for GDM with IMPRs

In this subsection, we will compare our approach with the existing approach for GDM with IMPRs and demonstrate the advantage of the proposed approach. Generally speaking, a common approach for GDM with IMPRs involves the following steps:

Algorithm 6.1.

Step 1: Aggregate the individual IMPRs into the collective IMPR.
Step 2: Aggregate the preference values of each alternative in the collective IMPR, and derive the overall preference value of each alternative.
Step 3: Rank all the alternatives and select the best one in accordance with the overall preference values.

The difference between the algorithms 5.1 and 6.1 is that the former first aggregates the different opinions provided by the DMs for a paired comparison of alternatives and derives the collective interval multiplicative preference information, while the latter eliminates step 1 in the former, i.e., does not perform such an aggregation, and directly collects the individual interval multiplicative preference information into the interval-valued hesitant multiplicative preference information. In the following, a concrete example is given to compare the results of the rankings of alternatives obtained with two approaches.

Example 6.2. Let's revisit Example 6.1 using Algorithm 6.1.

Step 1: Utilize the geometric averaging (GA) operator to aggregate the individual DMs' interval multiplicative preference opinions with respect to the attribute c_k ($k=1,2,3$) into the collective interval multiplicative preference opinion with respect to the attribute c_k, which are shown in Tables 6-8.

Step 2: Calculate the overall interval multiplicative preference value $r_i^{(k)}$ of the alternative x_i with respect to the attribute c_k by the geometric averaging (GA) operator (see Table 9). For example,

$$\begin{aligned} r_2^{(1)} &= \mathrm{GA}\left(r_{21}^{(1)}, r_{22}^{(1)}, r_{23}^{(1)}, r_{24}^{(1)}\right) \\ &= \mathrm{GA}\left([0.2236, 0.2887],\ [1,1],\ [3,\ 4.4721],\ [1/2,\ 1]\right) \\ &= \left\{[0.8612, 1.0534], [0.8915, 1.1147], [0.8915, 1.2724]\right\} \end{aligned}$$

Step 3: Utilize the weighted geometric averaging (WGA) operator to aggregate all $r_i^{(k)}$ ($k=1,2,3$) into an overall r_i of the alternative x_i:

$$r_1 = \mathrm{WGA}\left(r_1^{(1)}, r_1^{(2)}, r_1^{(3)}\right) = \mathrm{WGA}\left([1.2449, 1.7201], [0.9306, 1.2676], [0.4201, 0.5512]\right) = [0.7770, 1.0495]$$

$$r_2 = [1.0402, 1.3694], \quad r_3 = [1.0864, 1.4279], \quad r_4 = [0.6338, 0.8755].$$

Step 4: In accordance with r_i ($i=1,2,3,4$), the ranking of the four alternatives is $x_3 > x_2 > x_1 > x_4$ and the best choice is x_3.

Table 6. The collective IMPR $R^{(1)}$ with respect to the criterion c_1.

6	x_1	x_2	x_3	x_4
x_1	[1,1]	[3.4641, 4.4721]	[1/3, 1/2]	[2.0801, 3.9149]
x_2	[0.2236, 0.2887]	[1,1]	[3, 4.4721]	[1/2, 1]
x_3	[2, 3]	[0.2236, 1/3]	[1,1]	[0.4082, 1]
x_4	[0.2554, 0.4807]	[1, 2]	[1, 2.4498]	[1,1]

Table 7. The collective IMPR $R^{(2)}$ with respect to the criterion c_2.

7	x_1	x_2	x_3	x_4
x_1	[1,1]	[1, 1.7321]	[1/4, 1/3]	[3, 4.4721]
x_2	[0.5773, 1]	[1,1]	[0.2924, 0.4368]	{[6, 7]
x_3	[3, 4]	[2.2894, 3.4200]	[1,1]	[0.1882, 0.2554]
x_4	[0.2236, 1/3]	[1/7, 1/6]	[3.9154, 5.3135]	[1,1]

8	x_1	x_2	x_3	x_4
x_1	[1,1]	[0.1543, 0.2236]	[1/9, 1/8]	[1.8171, 3.3019]
x_2	[4.4723, 6.4809]	[1,1]	[5.4772, 7]	[1/4, 1/3]
x_3	[8, 9]	[1/7, 0.1826]	[1,1]	[3, 5]
x_4	[0.3029, 0.5503]	[3, 4]	[1/5, 1/3]	[1,1]

Table 8. The collective IMPR $R^{(3)}$ with respect to the criterion c_3.

8	x_1	x_2	x_3	x_4
x_1	[1,1]	[0.1543, 0.2236]	[1/9, 1/8]	[1.8171, 3.3019]
x_2	[4.4723, 6.4809]	[1,1]	[5.4772, 7]	[1/4, 1/3]
x_3	[8, 9]	[1/7, 0.1826]	[1,1]	[3, 5]
x_4	[0.3029, 0.5503]	[3, 4]	[1/5, 1/3]	[1,1]

Table 9. The aggregation results of the alternatives x_i ($i = 1, 2, 3, 4$) with respect to the criteria c_k ($k = 1, 2, 3$).

9	c_1	c_2	c_3
x_1	[1.2449, 1.7201]	[0.9306, 1.2676]	[0.4201, 0.5512]
x_2	[0.7610, 1.0660]	[1.0032, 1.3223]	[1.5731, 1.9720]
x_3	[0.6536, 1.0000]	[1.0663, 1.3672]	[1.3607, 1.6931]
x_4	[0.7109, 1.2388]	[0.5947, 0.7371]	[0.6529, 0.9255]

From the above results, we can see that the ranking of alternatives obtained with Algorithm 6.1 is different from the one obtained with Algorithm 5.1. The reason for the difference is that Algorithm 6.1 first needs to aggregate the individual interval multiplicative preference values to the collect interval multiplicative preference value. In fact, such an aggregation is equivalent to transformation of the interval-valued hesitant multiplicative preference value into the interval multiplicative preference value, which may cause the loss of information. Contrary to Algorithm 6.1, Algorithm 5.1 does not need such an aggregation and therefore can preserve the original information as much as possible. As a consequence, the comparison results clearly illustrate the advantage of the proposed approach for MCDM based on IVHMPRs.

6.3. Comparison with the IVHPRs

According to Definitions 2.3 and 5.2, we can see that the difference between the IVHPRs and the IVHMPRs is that the former uses 0.1-0.9 scale, while the latter uses 1-9 scale. As stated before, the 1-9 scale may be more consistent with our intuition than the 0.1-0.9 scale in some cases. As a result, the IVHMPRs may be more appropriate to deal with some situations than the IVHPRs. An IVHPR $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$ can be transformed into an IVHMPR $\overline{R} = \left(\overline{r}_{ij}\right)_{n\times n}$ by the following equation:

$$\bar{r}_{ij} = \left\{ \left[9^{2\left(\tilde{r}_{ij}^{s}\right)^{L}-1}, 9^{2\left(\tilde{r}_{ij}^{s}\right)^{U}-1} \right] \middle| s = 1, 2, \cdots, l_{\tilde{r}_{ij}} \right\} \tag{53}$$

where $\tilde{r}_{ij} = \left\{ \tilde{r}_{ij}^{s} \middle| s = 1, 2, \cdots, l_{\tilde{r}_{ij}} \right\}$, and $i, j = 1, 2, \cdots, n$.

In what follows, we use an example adapted from [37] to show the difference between them.

Example 6.3 [37]. Let's revisit the example given in Section 6 of Ref. [24]. Suppose that $\tilde{R}^{(k)}$ ($k = 1, 2, 3$) are three IVHPRs given in [24] (see Tables 10-12).

Table 10. The IVHPR $\tilde{R}^{(1)}$

10	x_1	x_2	x_3	x_4
x_1	{[0.5, 0.5]}	{[0.4, 0.5], [0.7, 0.9]}	{[0.5, 0.6], [0.8, 0.9]}	{[0.3, 0.5]}
x_2	{[0.1, 0.3], [0.5, 0.6]}	{[0.5, 0.5]}	{[0.4, 0.5]}	{[0.6, 0.8]}
x_3	{[0.1, 0.2] , [0.4, 0.5]}	{[0.5, 0.6]}	{[0.5, 0.5]}	{[0.3, 0.4], [0.5, 0.6]}
x_4	{[0.5, 0.7]}	{[0.2, 0.4]}	{[0.4, 0.5], [0.6, 0.7]}	{[0.5, 0.5]}

Table 11. The IVHPR $\tilde{R}^{(2)}$

11	x_1	x_2	x_3	x_4
x_1	{[0.5, 0.5]}	{[0.2, 0.3], [0.5, 0.6]}	{[0.5, 0.6], [0.7, 0.9]}	{[0.2, 0.4]}
x_2	{[0.4, 0.5], [0.7, 0.8]}	{[0.5, 0.5]}	{[0.5, 0.8]}	{[0.3, 0.5], [0.6, 0.7], [0.8, 0.9]}
x_3	{[0.1, 0.3], [0.4, 0.5]}	{[0.2, 0.5]}	{[0.5, 0.5]}	{[0.4, 0.5], [0.7, 0.8]}
x_4	{[0.6, 0.8]}	{[0.1, 0.2], [0.3, 0.4], [0.5, 0.7]}	{[0.2, 0.3], [0.5, 0.6]}	{[0.5, 0.5]}

Table 12. The IVHPR $\tilde{R}^{(3)}$

12	x_1	x_2	x_3	x_4
x_1	{[0.5, 0.5]}	{[0.4, 0.5], [0.7, 0.8]}	{[0.6, 0.7]}	{[0.3, 0.5], [0.6, 0.7]}
x_2	{[0.2, 0.3], [0.5, 0.6]}	{[0.5, 0.5]}	{[0.4, 0.6]}	{[0.7, 0.8]}
x_3	{[0.3, 0.4]}	{[0.4, 0.6]}	{[0.5, 0.5]}	{[0.3, 0.4], [0.5, 0.7], [0.8, 0.9]}
x_4	{[0.3, 0.4], [0.5, 0.7]}	{[0.2, 0.3]}	{[0.1, 0.2], [0.3, 0.5], [0.6, 0.7]}	{[0.5, 0.5]}

First, we use Eq. (53) to transform the IVHPRs $\tilde{R}^{(k)}$ ($k=1,2,3$) into the IVHMPRs $\overline{R}^{(k)}$ ($k=1,2,3$), which are shown in Tables 13-15. Then, we use Algorithm 5.1 to derive the ranking of the alternatives as: $x_2 > x_1 > x_4 > x_3$, which is different from the one ($x_1 > x_2 > x_4 > x_3$) [24] obtained with the IVHPRs . The reason is that when describing the same preferences information, the IVHPR uses 0.1-0.9 scale, while the IVHMPR uses 1-9 scale. The IVHMPR is more consistent with our intuition than the IVHPR. Thus, our ranking may be more reasonable and reliable than the one obtained in [24].

Table 13. The IVHMPR $\overline{R}^{(1)}$

13	x_1	x_2	x_3	x_4
x_1	{[1, 1]}	{[0.6444, 1], [2.4082, 5.7995]}	{[1, 1.5518], [3.7372, 5.7995]}	{[0.4152, 1]}
x_2	{[0.1724, 0.4152], [1, 1.5518]}	{[1, 1]}	{[0.6444, 1]}	{[1.5518, 3.7372]}
x_3	{[0.1724, 0.2676] , [0.0.6444, 1]}	{[1, 1.5518]}	{[1, 1]}	{[0.4152, 0.0.6444], [1, 1.5518]}
x_4	{[1, 2.4082]}	{[0.2676, 0.6444]}	{[0.0.6444, 1], [1.5518, 2.4082]}	{[1, 1]}

Table 14. The IVHMPR $\overline{R}^{(2)}$

14	x_1	x_2	x_3	x_4
x_1	{[1, 1]}	{[0.2676, 0.4152], [1, 1.5518]}	{[1, 1.5518], [2.4082, 5.7995]}	{[0.2676, 0.6444]}
x_2	{[0.6444, 1], [2.4082, 3.7372]}	{[1, 1]}	{[1, 3.7372]}	{[0.4152, 1], [1.5518, 2.4082], [3.7372, 5.7995]}
x_3	{[0.1724, 0.4152], [0.6444, 1]}	{[0.2676, 1]}	{[1, 1]}	{[0.6444, 1], [2.4082, 3.7372]}
x_4	{[1.5518, 3.7372]}	{[0.1724, 0.2676], [0.4152, 0.6444], [1, 2.4082]}	{[0.2676, 0.4152], [1, 1.5518]}	{[1, 1]}

Table 15. The IVHMPR $\overline{R}^{(3)}$

15	x_1	x_2	x_3	x_4
x_1	{[1, 1]}	{[0.6444, 1], [2.4082, 3.7372]}	{[1.5518, 2.4082]}	{[0.4152, 1], [1.5518, 2.4082]}
x_2	{[0.2676, 0.4152], [1, 1.5518]}	{[1, 1]}	{[0.6444, 1.5518]}	{[2.4082, 3.7372]}
x_3	{[0.4152, 0.6444]}	{[0.6444, 1.5518]}	{[1, 1]}	{[0.4152, 0.6444], [1, 2.4082], [3.7372, 5.7995]}
x_4	{[0.4152, 0.6444], [1, 2.4082]}	{[0.2676, 0.4152]}	{[0.1, 0.2676], [0.4152, 1], [1.5518, 2.4082]}	{[1, 1]}

7. Conclusions

In this paper, we have defined the concept of IVHMSs by replacing the 0.1–0.9 scale in the IVHFSs by the 1–9 scale. We have proposed some fundamental operational laws on the IVHMSs and developed several operators for aggregation the interval-valued hesitant multiplicative information, including the generalized interval-valued hesitant multiplicative weighted averaging (GIVHMWA) operator, the interval-valued hesitant multiplicative weighted averaging (IVHMWA) operator, the generalized interval-valued hesitant multiplicative weighted geometric (GIVHMWG) operator and the interval-valued hesitant multiplicative weighted geometric (IVHMWG) operator. Some interesting properties and special cases have also been discussed. Considering that the decision makers provide some possible interval multiplicative preference values when they compare two alternatives, we have further defined the IVHMPR, which collects all the possible interval multiplicative preference values into an IVHME as its basic element. Moreover, an approach for MCDM with the IVHMPRs has been developed. In the end, we have compared the IVHMPRs with the IVHPRs and IMPRs by some numerical examples, and illustrated the advantages of the developed IVHMPRs over the IVHPRs and IMPRs.

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Authors' Contributions

'Zhiming Zhang' designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. 'Chong Wu' managed the analyses of the study and the literature searches. All authors read and approved the final manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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