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# Partial Groups, Simplicial $K(G, 1)$'s and Kan Complexes

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## Abstract

In our paper Simplicial $K(G, 1)$'s we constructed a sub-complex of the nerve of a group $G$ determined by a partial group structure, and we proved, under a generalized associativity condition called regularity, that the sub-complex realizes as a $K(G, 1)$. This type of sub-complex appears naturally in several topological and algebraic contexts. In this note we prove that regularity of a partial group implies that the Kan extension condition is satisfied on its nerve in dimensions greater than one, and in dimension one a weaker version of the extension condition holds.

## Keywords

Partial Group, Simplicial Set, Nerve, Kan Extension

## 1. Introduction

In our paper Simplicial $K(G, 1)$'s [1], we constructed a sub-complex of the nerve of a group $G$ determined by a partial group structure, and we proved, under a generalized associativity condition called regularity that the sub-complex realizes as a $K(G, 1)$. The purpose of this note is to prove (Theorem 3.5) that regularity implies that the classical Kan extension condition is satisfied on its nerve, but only in the following weaker sense. The extension condition can be defined in each dimension separately, and in the case of a partial group regularity implies that the extension condition is satisfied in dimensions 2 and higher, and in dimension 1 a weaker extension condition holds. The converse is not true. Implications of the result are discussed at the conclusion of the article. The motivation for our work comes primarily from two topological contexts where partial groups occur naturally: co-dimension 1 real analytic foliations, whose classifying space is the realization of the nerve of a partial group [2], and the discrete group of homeomorphisms of the circle where the partial group exhibits the discrete Euler class [3]. Other applications are discussed in [4], and in Section 4. By comparing regularity to the extension condition an objective is to bring the constructions into a more classical setting, and thereby enhance our understanding of the applications as well as encourage the search for more examples.

Concerning the terminology "partial group": in our paper [1] the starting point was a group $G$ presented by generators with relations coming from a binary operation satisfying a condition that we called *regularity*. In particular, we were interested in describing the homotopy type of the classifying space for real analytic co-dimension 1 foliations, [2]. Only later did we come to focus on the relations themselves and call a set of generators with such a binary operation a *partial group* [5]. In [6], pp.106-107, partial groups were referred to as group-like and arise from partial actions of groups. Independently partial groups were introduced in [7] in connection with Fusion Systems.

## 2. Partial Groups and *K*(*G*, 1)'s

**Definition 2.1.** A set $P$ is a *partial group* if associated to each pair $(f, g) \in P \times P$ there is at most one product element $fg \in P$ so that the following conditions are satisfied.

1) There exists an element $1 \in P$ satisfying $f1 = 1f = f$ for each $f \in P$.

2) For each $f \in P$ there exists an element $f^{-1} \in P$ so that $ff^{-1} = f^{-1}f = 1$.

3) If $fg = h$ is defined then so is $g^{-1}f^{-1} = h^{-1}$.

Naturally associated to partial groups are simplicial constructions which we now describe. Let $P_k = P \times \cdots \times P$, $k$ times, and $\mathcal{P} = \bigcup_k P_k$. Let an arrow $\rightarrow$ denote the transitive relation on $\mathcal{P}$ generated by $(f_1, \cdots, f_k) \in P_k$ is related to $(f_1, \cdots, f_i f_{i+1}, \cdots, f_k) \in P^{k-1}$ when $f_i f_{i+1}$ is defined. We refer to $(\mathcal{P}, \rightarrow)$ as the *arrow diagram associated to P.*

**Definition 2.2.** A partial group $P$ is said to be *regular* if the following is satisfied. Let $f, s, t \in \mathcal{P}$. If $f \rightarrow s$, $s \in P$, and $f \rightarrow t$ where $t$ is minimal with respect to $\rightarrow$, then $s = t$.

The regularity condition says that if some sequence of multiplications *reduces* a given $k$-tuple of elements to an element of the partial group, then no other way of composing the entries can terminate until a single, and necessarily unique, element is attained. Critical features of multiplication in a regular partial group appear in the main examples. Given $f, g, h \in P$ with $fg$ and $gh \in P$, it does not follow, in general, that $(fg)h \in P$ nor that $f(gh) \in P$. Moreover, it's possible that $f(gh)$ is defined, whereas $fg$ is not. Effectively regularity compensates for the failure of associativity. Each partial group $P$ has a simplicial nerve $NP$ and a classifying space $BP$ which is the realization $|NP|$. We recall one such construction for a partial group which is a group $G$, and then generalize to all partial groups. Let $N_q G = G^q$, and take $G^0 = 1$. For $q = 1$ the two face maps $N_1 G \rightarrow N_0 G$ are constant. For $q \geq 1$ the face maps $N_{q+1} G \rightarrow N_q G$ are $\partial_0, \cdots, \partial_{q+1}$ where

$$\partial_0(f_0,\cdots,f_q)=(f_1,\cdots,f_q),\ \partial_{q+1}(f_0,\cdots,f_q)=(f_0,\cdots,f_{q-1}),$$

and otherwise $\partial_i(f_0,\cdots,f_q)=(f_0,\cdots,f_{i-1}f_i,\cdots,f_q)$.

Degeneracies are $s_i:N_qG\to N_{q+1}G$ where 1 is inserted in the *i*-th place.

$$s_i(f_0,\cdots,f_{q-1})=(f_0,\cdots,1,\cdots,f_{q-1}).$$

The nerve realizes as the classifying space *BG* of the group *G*, and *BG* is a $K(G,1)$. We now generalize to partial groups. Let $N_0P=\{1\}$, $N_1P=P$ and define inductively

$$N_{q+1}P=\{(f_0,\cdots,f_q)\mid\partial_i(f_0,\cdots,f_q)\in N_qP,0\le i\le q+1\}$$

where the $\partial_i$ and $s_i$ are defined as for *NG* above. So, the set of $q+1$-simplices consists of those $q+1$-tuples of elements of *P* that reduce to a unique element of *P* with *all orders of operations*. We refer to this condition as *full associativity*, and call any $q+1$-tuple with this property *fully associative*. The *universal group of P*, denoted $\hat{P}$, is the free group on *P* modulo the relation $f\cdot g=fg$ whenever $fg$ is defined in *P*. Here $f\cdot g$ denotes the product in the free group. $\hat{P}$ is the fundamental group of *BP*. The formal inverse of an element in $\hat{P}$ is identified with its inverse in *P*, and the identity in $\hat{P}$ is identified with the identity in *P*. The vertices of the arrow diagram correspond to unreduced words in the elements of *P*, and each arrow corresponds to a reduction of the word to one of smaller length. The elements of $\hat{P}$ correspond to equivalence classes of elements of *P* under the equivalence relation $v_1\sim v_2$ if there exist *v* so that $v\to v_1$ and $v\to v_2$.

The following statement is verified in [1]. (It is however an easy consequence of the definitions.) In a regular partial group *P* every word other than 1 which is minimal in the arrow diagram represents a non-trivial element in $\hat{P}$ and every word representing 1 in the universal group reduces to 1 under the arrow relation. (This solves the identity problem in $\hat{P}$.) The main result of [1] is the following.

**Theorem 2.3.** *The classifying space BP is a* $K(\hat{P},1)$ *whenever P is regular.*

## 3. Partial Groups and the Extension Condition

**Definition 3.1.** A simplicial set is said to satisfy the extension condition if for every collection of $q+1$ *q*-simplices $x_0,\cdots,x_{k-1},x_{k+1},\cdots,x_{q+1}$ which satisfy the compatibility condition $\partial_ix_j=\partial_{j-1}x_i$, $i<j$, $i\ne k$. $j\ne k$, there exists a $(q+1)$-simplex *x* such that $\partial_ix=x_i$ for $i\ne k$. A simplicial set which satisfies the extension condition is called a Kan complex [8]. See [8] as well for all basic definitions and constructions involving simplicial sets.

The nerve of a partial group is a Kan complex only when the partial group is a group, for otherwise $f,g\in P$ does not imply $fg\in P$ in general. However we will see that the nerve of a regular partial group satisfies a weaker version of the extension condition for 1-simplices.

**Definition 3.2.** A simplicial set satisfies the extension condition in dimension *r* if the extension condition holds for every collection of $r+1$ *r*-simplices satisfying compatibility.

**Definition 3.3.** A simplicial set satisfies the weak extension condition in dimension 1 if for every trivial loop $y = y_0 y_1 \cdots y_m$ in $\pi_1(K,\phi)$ there exists a pair of consecutive 1-simplices $y_i, y_{i+1}$, $i$ mod ($m$ + 1), which satisfy the extension condition in dimension 1.

**Remark 3.4.** After replacing $y_i \cdot y_{i+1}$ by the "missing face" a trivial loop of smaller length, homotopic to $y$ is obtained. The process can be continued until the constant degenerate loop is obtained.

The following is our main result.

**Theorem 3.5.** *If a partial group P is regular then its nerve $N_q P$ satisfies the extension condition for all* $q \geq 2$, *and the weak extension condition in dimension* $q = 1$.

We begin with a lemma which adapts compatibility to our context.

**Lemma 3.6.** *Let* $\{x_0, \cdots, \hat{x}_k, \cdots, x_{q+1}\}$ *be a collection of* $q+1$ *q-simplices of* $N_q P$ *satisfying the compatibility condition. The there exists a* $(q+1)$*-tuple* $f = (f_0, \cdots, f_q)$ *of elements of P so that* $x_i$ *has the form* $x_i = \partial_i f$ *for all i,* $0 \leq i \leq q+1$.

*Proof.* Note the Lemma does not claim that $f$ is in $N_{q+1}P$. It will require the regularity condition to reach that conclusion. Suppose first that $0 < k < q+1$, and that $x_{q+1} = (f_0, \cdots, f_{q-1})$. Then $x_0 = (f_1, \cdots, f_{q-1}, f_q)$ for some $f_q \in P$, and that determines $(f_0, \cdots, f_q)$. If $k = 0$ again let $x_{q+1} = (f_0, \cdots, f_{q-1})$. Then $x_q = (f_0, \cdots, f_{q-1} f_q)$, and that determines $(f_0, \cdots, f_q)$. Note that the face maps determine the "missing face" $x_k$ of the compatible collection as well as all the other faces. There is therefore one and only one $f$ so that $\partial_i f = x_i$ for all $i$. □

Returning to a consideration of the Theorem, and applying the previous Lemma, we must verify that whenever each of $x_0, \cdots, \widehat{x_k}, \cdots, x_{q+1}$ is in $N_q P$ it follows, under the condition of regularity, that $x_k$ is in $NP$ as well. For then the definition of $NP$ implies that $(f_0, \cdots, f_q)$ is in $N_{q+1}P$. Now recall a $q$-tuple of elements of $P$ is a $q$-simplex of $N_q P$ if and only if it is fully associative. We are therefore reduced to verifying that when a particular $x_k$ is excluded from the set of $x_i$ and all the remaining $x_i$ are fully associative, then $x_k$ is fully associative as well. Before proving the general case we check directly that regularity implies the extension condition when $q = 2$. Consider $x_0 = (g_1, g_2)$ $x_1 = (g_0 g_1, g_2)$ $x_2 = (g_0, g_1 g_2)$ $x_3 = (g_0, g_1)$ Whichever $x_i$ is excluded, note that the initial products $g_0 g_1$ and $g_1 g_2$ are defined. If $k = 1$ then $x_2 \in NP$ implies $g_0(g_1 g_2)$ is defined. By regularity if $g_0 g_1$ is defined then $(g_0 g_1) g_2$ must be defined as well, hence $x_1$ is fully associative. The case $k = 2$ is essentially identical, and when $k = 0$ or $k = 3$ the verification is trivial.

*Proof.* For the general case of Theorem 3.5, regardless of which $x_k$ is excluded, every initial product $g_{i-1} g_i$ is defined. For the sake of definiteness, let us assume $k = 1$. The verification for $0 < k < q+1$ is the same, and for $k = 0$ and $k = q+1$ is trivial. We claim that $(g_0 g_1, \cdots, g_{q+1})$ is fully associative. Now $g_{i-1} g_i$ is defined. Assume first that $3 \leq i \leq q+1$, and consider

$(g_0g_1,\cdots,g_{i-1}g_i,\cdots,g_{q+1})$. Since this simplex can also be obtained by first applying the product $g_{i-1}g_i$ to $x_i$ and then $g_0g_1$ the simplex $(g_0g_1,\cdots,g_{i-1}g_i,\cdots,g_{q+1})$ is fully associative. Hence by regularity $(g_0g_1,\cdots,g_{i-1},g_i,\cdots,g_{q+1})$ is fully associative, as claimed. The only remaining non-trivial case to verify is that $((g_0g_1)g_2,\cdots,g_{q+1})$ is fully associative. But, by the case $q=2$ that simplex is also the reduction of $(g_0,g_1g_2,\cdots,g_{q+1})$ which is fully associative. This completes the proof of Theorem 3.5. □

We've observed when $q=1$, despite regularity, the extension property fails unless the partial group is a group. The simplest example of a partial group which is not a group and satisfies regularity is the generating set of a free group. The only allowable products are the trivial ones—that is those required by definition.

## 4. Remarks

### 4.1. Algebra

Group-like structures have an extensive history and have played a role in a variety of fields, [9]. Some examples like monoids and groupoids fit into a categorical setting, others like semigroups and quasigroups do not. At the time of [1] and [2] the closest construction appearing in the literature was that of pregroup, defined by J. Stallings [10], in his work on 3-manifolds. Pregroups are regular partial groups, but not conversely. The regular partial group $\pi_0(\Gamma_1^\omega)$ provides a counterexample [2]. More recently pregroups have been generalized [11], and the theory has evolved in various ways. In [6] partial group actions are defined and applied. We mention three papers which involve partial groups, fusion systems and homotopy theory: [7] [12] [13]. For a more formal algebraic treatment of partial groups see [14]. The converse of Theorem 3.5 does not hold. That is to be expected since by definition the simplices of the nerve of a partial group are fully associative, whereas the condition that has to be satisfied for regularity requires testing arbitrary $n$-tuples. So to find a counterexample it is only necessary to produce a partial group which is not regular. Note that in a partial group a product $f\circ g$, other than one that is trivially required by the axioms, can be removed, along with its inverse, from the set of allowable products and the partial group structure is maintained. Choose a partial group $P$ that is "big enough", in the sense that there are elements $f,g,h,k$ which together with their inverses are all distinct, different from the identity, and no products are trivial. This can be done, for example, in $\pi_0(\Gamma_1^\omega)$ which is uncountable. Assume furthermore that $f$, $f\circ g$, $(f\circ g)\circ h$, $((f\circ g)\circ h)\circ k$, and $g\circ h$ are all defined. Remove $f\circ(g\circ h)$ and $(g\circ h)\circ k$. Then the partial group $\pi_0$ is no longer regular for $(f,g,h,k)\to t$ for some $t\in P$, but $(f,g\circ h,k)$ is minimal.

### 4.2. Homotopy

As follows from the general theory of Kan complexes when the extension condition is satisfied it can be used to define homotopy groups intrinsically and sim-

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plicially, and the results agree with those defined topologically by way of their realizations ([8], see p. 7). The nerve $N\hat{P}$ is a *completion to extendability of the compatibility condition* on *NP*.

**Definition 4.1.** Let $X$ be a simplicial set with one vertex. The *e*-completion $Y$ of $X$ is defined as follows For every horn of 1-simplices which doesn't already satisfy the extension condition attach a 2-simplex so that the Kan extension condition is satisfied in dimension 1. Now consider horns of 2-simplices which do not already satisfy the Kan extension condition and attach 3-simplices. Since the Kan extension condition is satisfied in dimension 2 for simplices which are in $X$ no 3-simplices are attached to $X$. Continue attaching simplices in this manner to obtain $Y$ which satisfies the extension conditions in all dimensions without changing $X$ in dimensions greater than 1.

**Theorem 4.2.** *Suppose X satisfies the Kan extension condition in dimensions greater than or equal to 2. Let Y be the e-completion of X. Then the inclusion* $X \to Y$ *induces a monomorphism* $\pi_n(|X|) \to \pi_n(|Y|)$ *for all* $n \geq 2$, *and an epimorphism for* $n = 1$. *If, in addition, X satisfies the weak extension condition in dimension 1 then* $\pi_1(|X|) \to \pi_1(|Y|)$ *is an isomorphism.*

*Proof.* Because $|Y|$ is a Kan complex topological and simplicial homotopy class agree. Consider a homotopy class in $|X|$ which is trivial in $|Y|$. In $|Y|$ the topological class is represented by a simplicial homotopy class and, by the definition of simplicial homotopy, if it bounds then it bounds simplicially in $|X|$. This shows that the induced homomorphism on homotopy groups is a monomorphism for all $n \geq 2$.

In dimension 1, by construction of $Y$, every 1-simplex in $Y$ is a product of 1-simplices of $X$ so that every loop of 1-simplices representing an element of $\pi_1(|Y|)$ is homotopic to a loop in $|X|$, showing that $\pi_1(|X|) \to \pi_1(|Y|)$ is onto. Now, by definition of the weak extension condition, any loop in $X$ representing a trivial loop in $Y$ bounds in $X$ so that $\pi_1(|X|) \to \pi_1(|Y|)$ is one to one.

For example, when $X$ is the nerve of a partial group satisfying the regularity condition the simplicial set $Y$ is the nerve of the universal group of the partial group, and the Theorem implies $X \to Y$ induces a homotopy equivalence on realizations. Theorem 2.4 is a consequence. Perhaps one can check directly, in specific cases, for example when the partial group is the one arising from the classification of co-dimension 1 real analytic foliations, that the hypotheses of Theorem 4.2 are satisfied without deducing the result from regularity.

### 4.3. Applications

Further applications are discussed in [4] and reveal an intriguing confluence of themes. The word problem, real analytic Γ-structures, the non-existence of co-dimension 1 real analytic foliations on spheres, the Poincare-Bendixson Theorem, the discrete Euler class, and Fusion Systems are all connected by results involving partial groups, simplicial $K(G, 1)$'s and, as we have observed here, the extension condition.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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