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Existence of Positive Solutions for a Fourth-Order Three-Point BVP with Sign-Changing Green's Function

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Abstract

In this article, by using a fixed point theorem, we study following fourth-order three-point BVP: $\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), & t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, \\ \alpha u(0) + u'''(\eta) = 0 \end{cases}$ where $f \in C([0,1] \times [0,+\infty), [0,+\infty))$ $\alpha \in [0,6)$ and $\eta \in \left[\frac{1}{2}, 1\right)$. The main point to emphasize is that although the corresponding Green's function is changing signs, by applying the fixed point theorem, we can still obtain at least two positive solutions and degreased solutions under certain suitable conditions.

Keywords

Fourth-Order Fourth-Point Boundary Value Problem, Function, Positive Solution, Existence, Fixed Point Greens Function

1. Introduction

The boundary value problem of fourth-order ordinary differential equations (BVP for short) has attracted much attention due to its amazing application in engineering, physics, material mechanics, fluid mechanics and so on. Many authors use Banach contraction to study the existence of single or multiple positive solutions for certain third-order BVP-Guo (Orem), Guo-Krasnoselsky (Krasnoselsky) Fixed point theorem, Leray-Schauder nonlinear substitution, fixed point index theory of viewing cone, monotonic iterative technique, upper and lower solution method, degree theory, the Critical point theorem in a conical shell, etc. see [1]-[6].

However, it is necessary to point out that, in most of the existing literature, the Greens functions involved are nonnegative, which is an important condition in the study on BVP Positive Solution.

Recently, when the corresponding Green's function is changing signs, some work has been done on the positive solution of the second or third order BVP. For example, Zhong and An [7] studied the existence of at least one positive solution of the following second-order periodic BVP with positive and negative transformation Green's function

$$\begin{cases} u^{(3)}(t)+\rho^{(2)}u=\lambda f(u), & 0<t<T, \\ u(0)=u(T), \\ u'(0)=u'(T). \end{cases}$$

where $0<\rho\le\frac{3\Pi}{2T}$. The main tool used is the fixed point index theory of cone mapping 2008, for a singular third-order three-point BVP of Green's function with infinite signature

$$\begin{cases} u^{(3)}(t)=a(t)f(t,u(t)), & 0<t<1, \\ u(0)=u(1)=u''(\eta)=0. \end{cases}$$

where $\eta\in\left(\frac{17}{24},1\right)$. Palamide and Smirlis [8] discussed the existence of at least one positive solution. Their technique is a combination of Guo-Krasnosel'sski fixed point theory and the corresponding vector field characteristics. In 2012, Sun and Zhao [9] [10] obtained single or multiple positive solutions with three-point positive and negative BVP by applying the fixed point theory of Guo-Krasnosel'skii and Leggett-Williams.

$$\begin{cases} u^{(3)}(t)=f(t,u(t)), & 0<t<1, \\ u(0)=u(1)=u''(\eta)=0. \end{cases}$$

where $\eta\in\left(\frac{1}{2},1\right)$. For relevant results, one can refer to [11]-[18]. It is worth mentioning that there are other types of achievements on either sign-changing or vanishing Green's functions which prove the existence of sign-changing solutions, positive in some cases, see [11] [19] [20] [21] [22].

Inspired and inspired by the above works, this article focuses on the following fourth-order three-point BVP with the iconic Green's function.

$$\begin{cases} u^{(4)}(t)=\lambda f(t,u(t)), & t\in[0,1], \\ u'(0)=u''=u(1)=0, \\ \alpha u(0)+u'''(\eta)=0 \end{cases}$$

Throughout this paper, we always assume that $\alpha\in[0,6)$ and $\eta\in\left[\frac{1}{2},1\right)$. Obviously, the BVP (2.1) is a special case of the BVP (2.2). However, it is necessary to point out that this paper is not a simple extension of [23], which is different from the restriction in [23]. On the other hand, compared with [23], we can only prove that the obtained solution is concave on $[0,\eta]$.

Our main tool is the following well-known Guo-Krasnoselskii fixed point theorem [24] [25]:

Let K be a cone in a real Banach space E.

Definition 1.1. A functional $\rho: K \to R$ is said to be increasing on K provided $\rho(x) \le \rho(y)$ for all $x, y \in K$ with $x \le y$, where $x \le y$ if and only if $y - x \in K$.

Definition 1.2. Let $\phi: K \to [0,+\infty)$ be continuous. For each $d > 0$, one defines the set

$$K(\phi,d) = \{u \in K : \phi(u) < d\}$$

Theorem 1.1. Let ρ and ϕ be increasing, nonnegative, and continuous functionals on K, and let φ be a nonnegative continuous functional on K with $\varphi(0) = 0$ such that, for some $c > 0$ and $M > 0$,

$$\phi(u) \le \varphi(u) \le \rho(u), \|u\| \le M\phi(u)$$

for all $u \in K(\phi,c)$. Suppose there exist a completely continuous operator $T: K(\phi,c) \to K$ and $0 < a < b < c$ such that

$$\varphi(\xi u) \le \xi\varphi(u) \text{ for } 0 \le \xi \le 1,\ u \in \partial K(\varphi,b),$$

and

(H1) $\phi(Tu) > c$ for all $u \in \partial K(\phi,c)$;

(H1) $\varphi(Tu) < bc$ for all $u \in \partial K(\varphi,b)$;

(H3) $K(\rho,a) \ne 0$ and $\rho(Tu) > a$ for all $u \in \partial K(\rho,a)$.

Then T has at least two fixed points u^* and u^{**} in $K(\phi,c)$ such that

$$a < \rho(u^*) \text{ with } \varphi(u^*) < b,$$

$$b < \varphi(u^{**}) \text{ with } \phi(u^{**}) < c.$$

2. Preliminaries

The remainder of this paper, we assume that Banach space $C[0,1]$ is equipped with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

For the following BVP:

$$\begin{cases} u^{(4)}(t) = \lambda f(t,u(t)), & t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, \\ \alpha u(0) + u'''(\eta) = 0 \end{cases} \tag{2.1}$$

then we have the following lemma.

Lemma 2.1. The BVP (2.1) has only trivial solution.

Proof. Easy to check.

Now, for any $y \in C[0,1]$, we consider the boundary value problems

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u(t)), \quad t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, \\ \alpha u(0) + u'''(\eta) = 0 \end{cases} \tag{2.2}$$

After a direct computation, one may obtain the expression of Green's function $G(t,s)$ of the BVP (2) as following:

For $s \ge \eta$

$$G(t,s) = \begin{cases} \dfrac{-(1-\alpha t^3)(1-s)^3}{6(6-\alpha)} & 0 \le t \le s, \\ \dfrac{(t-s)^3}{6} - \dfrac{(1-\alpha t^3)(1-s)^3}{6(6-\alpha)} & s \le t \le 1 \end{cases}$$

and $s < \eta$

$$G(t,s) = \begin{cases} \dfrac{6(1-t^3) - (1-\alpha t^3)(1-s)^3}{6(6-\alpha)} & 0 \le t \le s, \\ \dfrac{(t-s)^3}{6} + \dfrac{6(1-t^3) - (1-\alpha t^3)(1-s)^3}{6(6-\alpha)} & s \le t \le 1 \end{cases}$$

Lemma 2.2. It is not difficult to verify that $G(t,s)$ has the following characteristics:

1) If $s \in [1,\eta]$, then $G(t,s)$ is nonincreasing with respect to $t \in [0,1]$.

2) $G(t,s)$ changes its sign on $[0,1] \times [0,1]$. In details, if $(t,s) \in [0,1] \times [0,\eta]$, then $G(t,s) \ge 0$. If $(t,s) \in [0,1] \times [1,\eta]$, then $G(t,s) \le 0$.

3) If $s \ge \eta$, then $\max_{t\in[0,1]} G(t,s) = G(1,s) = 0$ such that

$$G(t,s) \ge 0 \quad \text{for} \quad 0 \le s \le \eta \quad \text{and} \quad G(t,s) \le 0 \quad \text{for} \quad \eta \le s \le 1.$$

Moreover, if $s \ge \eta$, then

$$\max G(t,s) : t \in [0,1] = G(1,s) = 0,$$

$$\min G(t,s) : t \in [0,1] = G(0,s) = \frac{-(1-s)^3}{6(6-\alpha)}$$

if $s < \eta$, then

$$\max G(t,s) : t \in [0,1] = G(0,s) = \frac{6-(1-s)^3}{(6-\alpha)},$$

$$\min G(t,s) : t \in [0,1] = G(1,s) = 0$$

Now, let $K_0 = \{y \in C[0,1] : y(t)\}$ is nonnegative and decreasing on $[0,1]$.

Then K_0 is a cone in C [0, 1].

Lemma 2.3. Let $y \in K_0$ and $u(t) = \int_0^1 G(t,s) y(s) \mathrm{d}s, t \in [0,1]$. Then u is the unique solution of the BVP (1.2) and $u \in K_0$. Moreover, $u(t)$ is concave on $[0,\eta]$.

Proof. For $0 \le t \le \eta$, we have

$$u(t)=\int_0^t\left[\frac{(t-s)^3}{6}+\frac{6(1-t^3)-(1-\alpha t^3)(1-s)^3}{6(6-\alpha)}\right]y(s)\mathrm{d}s$$
$$+\int_t^\eta\left[\frac{6(1-t^3)-(1-\alpha t^3)(1-s)^3}{6(6-\alpha)}\right]y(s)\mathrm{d}s$$
$$+\int_\eta^1\frac{-(1-\alpha t^3)(1-s)^3}{6(6-\alpha)}y(s)\mathrm{d}s$$

since $\eta\ge\frac{1}{2}$ we get

$$u'(t)=\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left[(1-s)^3-1\right]y(s)\mathrm{d}s+\frac{1}{2}\int_0^t\left(s^2-2ts\right)y(s)\mathrm{d}s$$
$$-\frac{t^2}{2}\int_t^\eta y(s)\mathrm{d}s+\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}y(s)\mathrm{d}s$$
$$\le y(\eta)\left[\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left[(1-s)^3-1\right]\mathrm{d}s+\frac{1}{2}\int_0^t\left(s^2-2ts\right)\mathrm{d}s\right.$$
$$\left.-\frac{t^2}{2}\int_t^\eta\mathrm{d}s+\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\mathrm{d}s\right]$$
$$\le t^2y(\eta)\left[\frac{\alpha(1-4\eta)}{8(6-\alpha)}-\frac{\eta}{2}+\frac{t}{6}\right]\le t^2y(\eta)\left[\frac{\alpha(1-4\eta)}{8(6-\alpha)}+\frac{\eta}{3}\right]\le 0$$

At the same time, $\eta>\frac{1}{2}$ shows that

$$u''(t)=\frac{\alpha t}{(6-\alpha)}\int_0^\eta\left[(1-s)^3-1\right]y(s)\mathrm{d}s-\int_0^t sy(s)\mathrm{d}s$$
$$-t\int_t^\eta y(s)\mathrm{d}s+\int_\eta^1\frac{\alpha t(1-s)^3}{(6-\alpha)}y(s)\mathrm{d}s$$
$$\le y(\eta)\left[\frac{\alpha t}{(6-\alpha)}\int_0^\eta\left[(1-s)^3-1\right]\mathrm{d}s-\int_0^t s\mathrm{d}s-t\int_t^\eta\mathrm{d}s+\int_\eta^1\frac{\alpha t(1-s)^3}{(6-\alpha)}\mathrm{d}s\right]$$
$$\le ty(\eta)\left[\frac{\alpha(1-4\eta)}{4(6-\alpha)}-\eta+\frac{t}{2}\right]\le ty(\eta)\left[\frac{\alpha(1-4\eta)}{4(6-\alpha)}-\frac{\eta}{2}\right]\le 0\qquad t\in(0,\eta)$$

For $t\in(\eta,1)$, we have

$$u(t)=\int_0^\eta\left[\frac{(t-s)^3}{6}+\frac{6(1-t^3)-(1-\alpha t^3)(1-s)^3}{6(6-\alpha)}\right]y(s)\mathrm{d}s$$
$$+\int_\eta^t\left[\frac{(t-s)^3}{6}-\frac{(1-\alpha t^3)(1-s)^3}{6(6-\alpha)}\right]y(s)\mathrm{d}s$$
$$+\int_t^1\left[\frac{-(1-\alpha t^3)(1-s)^3}{6(6-\alpha)}\right]y(s)\mathrm{d}s$$

In view of $y\in K_0$ and $\eta>\frac{1}{2}$, we get

$$
\begin{aligned}
u'(t) &= \frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left[(1-s)^3-1\right]y(s)\mathrm{d}s+\frac{1}{2}\int_0^\eta\left(s^2-2ts\right)y(s)\mathrm{d}s \\
&\quad +\int_\eta^t\frac{(t-s)^2}{2}+\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}y(s)\mathrm{d}s \\
&\le y(\eta)\left[\frac{\alpha t^2}{2(6-\alpha)}\int_0^\eta\left[(1-s)^3-1\right]\mathrm{d}s+\frac{1}{2}\int_0^\eta\left(s^2-2ts\right)\mathrm{d}s\right. \\
&\quad \left.+\int_\eta^t\frac{(t-s)^2}{2}+\int_\eta^1\frac{\alpha t^2(1-s)^3}{2(6-\alpha)}\mathrm{d}s\right] \\
&= t^2y(\eta)\left[\frac{-\alpha\eta}{2(6-\alpha)}+\frac{t}{6}-\frac{\eta}{2}\right]\le t^2y(\eta)\left[\frac{-\alpha\eta}{2(6-\alpha)}-\frac{\eta}{3}\right]\le 0 \quad t\in(\eta,1)
\end{aligned}
$$

Obviously, $u''''(t)=y(t)$ for $t\in[0,1]$, $u'(0)=u'''(0)=u(1)=0$, $\alpha u(0)+u''(\eta)=0$. This shows that u is a solution of the BVP (2.2). The uniqueness follows immediately from Lemma 2.1. Since $u'(t)\le 0$ for $t\in[0,1]$ and $u(1)=0$, we have $u(t)\ge 0$ for $t\in[0,1]$. So, $u\in K_0$. In view of $u''(t)\le 0$ for $t\in[0,\eta]$, we know that $u(t)$ is concave on $[0,\eta]$.

Lemma 2.4. Assume $y\in K_0$ then the unique solution $u(t)$ of the BVP (2.2) satisfies

$$\min_{t\in[0,\mu]}u(t)\ge\mu^*\|u\|$$

where $\mu\in\left(0,\frac{1}{2}\right]$ and $\mu^*=\frac{\eta-t}{\eta}$.

Proof. From Lemma 2.2, we know that $u(t)$ is concave on $[0,\eta]$, thus for $t\in[0,\eta]$,

$$u(t)\ge\frac{\eta-t}{\eta}u(0)+\frac{t}{\eta}u(\eta) \tag{2.3}$$

In view of $u\in K_0$, we know that $\|u\|=u(0)$, which together with (2.3) implies that

$$u(t)\ge\frac{\eta-t}{\eta}\|u\|,\quad 0\le t\le\eta$$

according to that

$$\min_{t\in[0,\mu]}u(t)=u(\mu)\ge\frac{\eta-\mu}{\eta}\|u\|=\mu^*\|u\|$$

3. Main Results

In this section, we are concerned with the existence of at least one positive solution of the problem (2.1). Assume that

(C1) For each $u\in[0,+\infty)$, the mapping $t\mapsto f(t,u)$ is decreasing;

(C2) For each $t\in[0,1]$, the mapping $u\mapsto f(t,u)$ is increasing.

Let

$$K=\left\{u\in K_0:\min_{t\in[0,\mu]}u(t)\ge\mu^*\|u\|\right\}$$

Then it is easy to see that K is a cone in $C[0,1]$.

Now, we define an operator $A:K\to K$ by

$$(Au)(t)=\int_0^1 G(t,s)\,f\left(s,u(s)\right)\mathrm{d}s, t\in[0,1]$$

distinctly, if u is a fixed point of A in K, then u is a positive and nondecreasing solution of the BVP (2.2), by lemma 2.3 and lemma 2.4 we know, $A:K\to K$ although $G(t,s)$ is not continuous, it follows from known textbook results, for example, see [26], that $A:K\to K$, is completely continuous. Set

$$\mu^*=\frac{\eta-\mu}{\eta},\quad P=\int_0^\eta\frac{1-(1-s)^3}{(6-\alpha)}\mathrm{d}s\quad \text{and}\quad Q=\int_0^\mu G(\eta,s)\mathrm{d}s$$

Lemma 3.1. Suppose that (C1) and (C2) hold. Moreover, If there exist three constants a, b and c with $0<a<b<\mu^*c$ such that

(F1) $f(\mu,c)>\dfrac{c}{P}$,

(F2) $f\left(0,\dfrac{b}{\mu^*}\right)<\dfrac{b}{Q}$,

(F3) $f(\mu,\mu^*a)>\dfrac{a}{P}$

then boundary value problem (1.1) has at least two positive solutions $u\in K$.

Proof. First, we define the increasing, nonnegative, and continuous functionals ϕ, φ and ρ on K as follows:

$$\phi(u)=\min_{t\in[0,\mu]}u(t)=u(\mu),$$

$$\varphi(u)=\max_{t\in[\mu,1]}u(t)=u(\mu),$$

$$\rho(u)=\max_{t\in[0,1]}u(t)=u(0).$$

Obviously, for any $u\in K$, $\phi(u)=\varphi(u)\le\rho(u)$. At the same time, for each $u\in K$, in view of $\phi(u)=\min_{t\in[0,\mu]}u(t)\ge\mu\in\|u\|$, we have

$$\|u\|\le\frac{1}{\mu^*}\phi(u)\quad \text{for } u\in K.$$

Furthermore, we also note that $\varphi(\xi u)=\xi\varphi(u)$ for $0\le\xi\le1$, $u\in K$.

Next, for any $u\in K$, we claim that

$$\int_\mu^1 G(\eta,s)\,f\left(s,u(s)\right)\mathrm{d}s\ge0 \tag{3.1}$$

In fact, it follows from (C1), (C2), and $\eta\ge\dfrac{\alpha+4}{16-\alpha}$

$$\begin{aligned}
&\int_\mu^1 G(\eta,s)\,f\left(s,u(s)\right)\mathrm{d}s\\
&=\int_\mu^\eta G(\eta,s)\,f\left(s,u(s)\right)\mathrm{d}s+\int_\eta^1 G(\eta,s)\,f\left(s,u(s)\right)\mathrm{d}s\\
&\le\int_\mu^\eta\left[\frac{(\eta-s)^3}{6}+\frac{6\left(1-\eta^3\right)-\left(1-\alpha\eta^3\right)(1-s)^3}{6(6-\alpha)}\right]y(s)\mathrm{d}s\\
&\quad+\int_\eta^1\frac{-\left(1-\alpha t^3\right)(1-s)^3}{6(6-\alpha)}\,f\left(s,u(s)\right)\mathrm{d}s
\end{aligned}$$

$$= f(\eta,u(\eta))\left[\int_{\mu}^{\eta}\frac{(\eta-s)^3}{6}+\frac{6(1-\eta^3)-(1-\alpha\eta^3)(1-s)^3}{6(6-\alpha)}\mathrm{d}s\right.$$
$$\left.+\int_{\eta}^{1}\frac{-(1-\alpha\eta^3)(1-s)^3}{6(6-\alpha)}\mathrm{d}s\right]$$
$$\geq\frac{f(\eta,u(\eta))}{24(6-\alpha)}\left[\alpha\eta^3(1-\eta)+24\eta-18\eta^4-1+\mu\left(36\eta^2+3\alpha\eta^3-6\alpha\eta^2-23\right)\right.$$
$$\left.+\mu^3\eta(4\alpha-24)+\mu^4(6-\alpha)\right]$$
$$=\frac{f(\eta,u(\eta))}{24(6-\alpha)}\left[\alpha\eta^3(1-\eta)+24\eta-18\eta^4+10\eta^2+\eta^3-\frac{20}{27}-\frac{697}{81}\right]\geq 0,$$

Now, we assert that $\phi(Au)>c$ for all $u\in\partial K(\phi,c)$. To prove this, let $u\in\partial K(\phi,c)$; that is, $u\in K$ and $\phi(u)=u(\mu)=c$. Then

$$u(t)\geq u(\mu)=c,t\in[0,\mu]. \tag{3.2}$$

Since $(Au)(t)$ is decreasing on $[0,1]$, it follows from (3.1), (3.2), (C2), (C1) and (F1) that

$$\begin{aligned}\phi(Au)&=(Au)(\mu)\geq(Au)(\eta)=\int_0^1 G(\eta,s)f(s,u(s))\mathrm{d}s\\&\geq\int_0^{\mu}G(\eta,s)f(s,u(s))\mathrm{d}s\geq\int_0^{\mu}G(\eta,s)f(\mu,c)\mathrm{d}s\\&>\frac{c}{P}\int_0^{\mu}G(\eta,s)\mathrm{d}s=c.\end{aligned}$$

Then, we assert that $\varphi(Au)<b$ for all $u\in\partial K(\varphi,b)$. To see this, suppose that $u\in\partial K(\varphi,b)$; that is, $u\in K$ and $\varphi(u)=b$. Since $\|u\|\leq\frac{1}{\mu^*}\phi(u)=\frac{1}{\mu^*}\varphi(u)$, we have

$$0\leq u(t)\leq\|u\|\leq\frac{b}{\mu^*},t\in[0,\eta]. \tag{3.3}$$

In view of the properties of $G(t,s)$, (F2), (3.3), (C1) and (C2), we get

$$\begin{aligned}\varphi(Au)&=(Au)=\int_0^1 G(\eta,s)f(s,u(s))\mathrm{d}s\leq\int_0^{\eta}G(\mu,s)f(s,u(s))\mathrm{d}s\\&\leq\int_0^{\eta}G(\mu,s)f\left(0,\frac{b}{\mu^*}\right)\mathrm{d}s<\frac{b}{Q}\int_0^{\eta}G(\mu,s)\mathrm{d}s=b.\end{aligned}$$

Finally, we assert that $K(\rho,a)\neq 0$ and $\rho(Au)>a$ for all $u\in\partial K(\rho,a)$. In fact, the constant function $\frac{a}{2}\in K(\rho,a)$. Moreover, for $u\in\partial K(\rho,a)$, that is $u\in K$ and $\rho(u)=u(0)=a$. Then

$$u(t)\geq\mu^*\|u\|=\mu^*u(0)=\mu^*a,t\in[0,\mu]. \tag{3.4}$$

Since $(Au)(t)$ is decreasing on $[0,1]$, it follows from (F3), (3.1), (3.4), (C1) and (C2) that

$$\rho(Au)=(Au)(0)\geq(Au)(\eta)=\int_0^1 G(\eta,s)f(s,u(s))\mathrm{d}s$$

$$\begin{aligned}&\geq \int_0^{\mu} G(\eta,s) f\left(s,u(s)\right)\mathrm{d}s \geq \int_0^{\mu} G(\eta,s) f\left(\mu,\mu^* a\right)\mathrm{d}s\\&> \frac{a}{P}\int_0^{\mu} G(\eta,s)\mathrm{d}s = a.\end{aligned}$$

To sum up, all the hypotheses of Theorem 1.1 are satisfied. Consequently A has at least two fixed points; that is, the BVP (1.1) has at least two positive solutions u^* and u^{**} such that

$$a < \max_{t\in[0,1]} u^*(t) \text{ with } \max_{t\in[\mu,1]} u^*(t) < b$$

$$b < \max_{t\in[\mu,1]} u^{**}(t) \text{ with } \min_{t\in[0,\mu]} u^{**}(t) < c.$$

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Conflicts of Interest

The author declares that there is no conflict of interest with the publication of this article.

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