Hindawi
Advances in High Energy Physics
Volume 2020, Article ID 7126374, 10 pages
https://doi.org/10.1155/2020/7126374

Research Article

Modified Anyonic Particle and Its Fundamental Gauge Symmetries

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Received 7 February 2020; Revised 16 April 2020; Accepted 23 April 2020; Published 22 May 2020

Academic Editor: Salvatore Mignemi

In this article, we study the possibility of changing a physical degree of freedom of a particle to its quantum spin after quantization is applied. Our approach to do such a survey is increasing the fundamental symmetries of the anyonic particle model with the help of the symplectic formalism of constrained systems. After extracting the corresponding Poisson structure of all constraints, we compare the effect of gauging on the phase spaces, the number of physical degrees of freedom, canonical structures of both primary and gauged models, and the spin of the anyon, in terms of its energy.

1. Introduction

It is quoted from the early days of discovering quantum mechanics and presenting the idea of the spin of the particle that if the intrinsic angular momentum (spin) of a point particle is created from its coordinate and momentum vectors, similar to the classical orbital angular momentum, then the speed of such a point particle exceeds the speed of light. Therefore, spin is chosen as an intrinsic, independent, and nonderivable vector of other types of degrees of freedom. Other investigations, especially in the field of particle physics, showed that the value of this angular momentum is measured as an integer or half integer of the fundamental constant $\hbar$, e.g., the spin of the electron, measured as $S_i = (1/2)\hbar\sigma_i$.

This angular momentum and its intrinsic quantum observables satisfy the commutating algebra $[S_i, S_j] = i\hbar\varepsilon_{ijk} S_k$. Accompanied with this theoretical rule of canonical quantization, which is presented later to quantize classical fields as $\{,\}_{DB} \rightarrow (1/i\hbar)[,]$, one can be classically intuited that we can have models, where the spin operator is obtained from the internal degrees of freedom (coordinate momentum).

This idea has the advantage of making the spin operator with any value (not only integer or half integer) for the physical models called anyons [1, 2]. The most famous model of this kind is the anyonic particles in condensed matter physics, which exist in the spaces with small dimensions [3].

Since we want all or a part of the coordinate-momentum vectors of the spin vector to be of nonphysical space-like vectors, we can write the initial particle model as a relativistic one [4, 5]. To do this, one should impose a nonphysical condition to the model. This can be done by adding a space-like vector as a nonphysical degree of freedom to a relativistic particle model, which will be discussed further. The spin operator is then obtained from the classical degree of freedom, after the canonical quantization is applied. Now, let us make such a relativistic particle model.

The prototype Lagrangian for a free relativistic particle with mass m can be considered as similar to its nonrelativistic congener, i.e., $L = (1/2)m\dot{x}^2$. Apparently, this Lagrangian does not apply any restriction on the particle to obey the second principle of relativity, i.e., having the velocity less than the speed of light. To remove such an ambiguity, we can use the Lagrangian of the free relativistic particle as

$$L = \sqrt{p^2(\dot{x}) + m^2} - m. \tag{1}$$

Although this Lagrangian satisfies the second law of relativity, it depends only on the momenta (phase-space coordinates) and must be transformed to the configuration space (i.e., to be a function of coordinates and corresponding velocities), using a proper map. Replacing the relativistic momentum with the four velocities of the particle, we obtain the appropriate Lagrangian for the free relativistic particle [6].

Apparently, Lagrangian (1) does not include any spin degree of freedom. In the classical mechanics point of view, this degree of freedom is not an observable quantity and will emerge after the model is quantized. A simple proposed model of a relativistic particle in $(d+1)$ dimensions $(\mu = 0, 1, \cdots, d-1)$ and with the metric $g_{\mu\nu} = \mathrm{diag}\,(+,-,\cdots,-)$ is described by the following Lagrangian [1]:

$$L = \frac{m(\dot{x}.\dot{n})}{\sqrt{\dot{n}^2}}, \tag{2}$$

where $\dot{n}$ is assumed to be time-like. It has been shown that the auxiliary space-like vector n^μ, satisfying condition $n^2 + 1 = 0$, classically adds the spin to the relativistic particle [1].

Tenably, Lagrangian (2) describes a particle with an arbitrary spin degree of freedom, which is called an anyon [1–3, 7–9]. Due to its interesting statistics, it has been used to model different phenomena in physics, such as the fractional quantum Hall effect and high-Tc superconductivity and even describing some physical processes in the presence of cosmic strings [1, 9–21].

This article is organized as follows. As we said, our goal is to make a gauged relativistic particle model with an arbitrary spin degree of freedom. Hence, in Section II, we review the constrained and symplectic structures of the primary non-gauged model by calculating Dirac brackets between phase-space variables. Using the symplectic embedding formalism of constrained systems, in Section III, we make a gauged particle model. As we will see, this procedure is only successful if we increase the phase space of the model by a couple of auxiliary variables, in addition to its usual coordinates and momenta and the nonphysical part $(n_\mu, \wp_\mu)$. Finally, we modify the nonphysical condition and get its momentum part as a light-like vector and obtain its effect on the spin of the anyon.

Section IV pursues the same goal as Section 2, but here, the Poisson structure of the gauged model is derived. To obtain the pure Poisson structure by second class constraints, we check the chain of constraints of the model to avoid any kind of bifurcation. At the end of this section, we obtain Dirac brackets of the gauged model.

Sections V and VI are presented in a way where the interested reader can check our claims and control our results. In Section V, we compare the physical degrees of freedom of both primary and gauged models. Gauge transformation generators of the model are also obtained in Section VI.

In Section VII, we analyse our results by solving an auxiliary coordinate and review its effect on the spin of anyons. This part is written in a proposing way, as a road map for further works.

2. Phase Space of the Anyonic Model

In order to see if the classical form of the spin degree of freedom contains a physical property or not, we try to increase the gauge symmetries of the Lagrangian (2). Our tool to do such an investigation is the symplectic embedding formalism of constrained systems [22–28].

Corresponding conjugate momenta of the coordinates x_μ and n_μ are

$$p_\mu = \frac{m\dot{n}_\mu}{\sqrt{\dot{n}^2}}, \tag{3}$$

$$\wp_\mu = \frac{m}{\sqrt{\dot{n}^2}}\left(\dot{x}_\mu - \frac{\dot{x}.\dot{n}}{\dot{n}^2}\dot{n}_\mu\right). \tag{4}$$

The space-like condition of the added spin vector imposes the following primary constraint as

$$\phi_0 := n^2 + 1 \approx 0, \tag{5}$$

which is recognised as a first-class constraint, afterwards.

Using the definition of momenta in (3) and the condition $n\dot{n} = 0$, we obtain the following null identities, depending on phase-space coordinates. These identities which are defined on the constraint surface are called primary constraints.

$$\phi_1 := p^2 - m^2 \approx 0, \tag{6}$$

$$\phi_2 := n \cdot p \approx 0, \tag{7}$$

$$\phi_3 := \wp \cdot p \approx 0, \tag{8}$$

where weak equality, ≈ 0, indicates the null nature of constraints on the constrained surface. Knowing this fact, we ignore using the weak equality for the rest of the article to show constraints.

Considering Dirac's classification of constraints, we see that ϕ_0 and ϕ_1 are classified as first-class constraints, i.e., their Poisson brackets and other constraints vanish, while ϕ_2 and ϕ_3 are called second-class ones [29], i.e., having nonzero Poisson brackets. The Poisson structure of the model is obtained by calculating Dirac brackets of phase-space variables, X_μ, via the following relation:

$$\{X_\alpha, X_\beta\}^* = \{X_\alpha, X_\beta\} - \{X_\alpha, \Phi_i\}\Delta_{ij}^{-1}\{\Phi_j, X_\beta\}, \tag{9}$$

where Φ_i is the set of all second-class constraints and Δ_{ij} is the Poisson bracket matrix of second-class constraints. Having $X_\alpha = (x_\mu, n_\mu, p_\mu, \wp_\mu)$, we calculate

$$\{x_\mu, x_\nu\}^* = \frac{1}{m^2}\left(\wp_\mu n_\nu - n_\mu \wp_\nu\right), \tag{10}$$

$$\{x_\mu, p_\nu\}^* = \delta_{\mu\nu}, \tag{11}$$

$$\{x_\mu, n_\nu\}^* = \frac{-1}{m^2}\left(n_\mu p_\nu\right), \tag{12}$$

$$\{x_\mu, \wp_\nu\}^* = \frac{-1}{m^2}\left(\wp_\mu p_\nu\right), \tag{13}$$

$$\{n_\mu, \wp_\nu\}^* = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}, \tag{14}$$

which in quantization leads to a noncommutative phase space [30]. It is worthwhile to mention that according to Dirac prescription [29], the quantized model, i.e., the Hilbert space of the quantum states, is fully available as

$$\{A, B\}^* \longrightarrow \frac{1}{i\hbar}[A, B]. \tag{15}$$

Apparently, the first-class constraint, ϕ_0, confirms the existence of a gauge symmetry in the model and persuades us to increase such a symmetry. Nevertheless, we need to check that this symmetry enhancement is only gained via n_μ or there would exist other degrees of freedom.

3. Gauging the Anyonic Model

In order to gauge the model via symplectic formalism, we must be assured that the canonical Hamiltonian exists. But for this model, H_c vanishes.

$$H_c = \dot{x}\cdot p + \dot{n}\cdot\wp - L = 0. \tag{16}$$

Thus, the embedding procedure which enhances the gauge symmetry is not applicable anymore. This happens due to the mixed nature of the model, consisting of both first- and second-class constraints. A preexisting second-class constraint in the model indicates the presence of a redundant degree of freedom which spoils the gauge symmetry [31].

To overcome this difficulty, we can use some auxiliary coordinates, such as conjugate variables (ξ_μ, π_μ), to extend the phase space of the original model and convert a mixed physical system to a pure second-class one, using the following extensions [32, 33]:

$$\begin{aligned} p_\mu &\longrightarrow p_\mu + \xi_\mu, \\ H_c &\longrightarrow H_c + \frac{1}{2}\pi^2. \end{aligned} \tag{17}$$

To transform the Lagrangian (2), one can do the following replacement:

$$L \longrightarrow L - \xi\cdot\dot{x} + \frac{1}{2}\dot{\xi}^2. \tag{18}$$

Since this replacement is a gauge-fixing term, which is inserted in the gauge invariant Lagrangian, the arbitrariness of the gauge-dependent variables will be destroyed, and via the variation of the new Lagrangian, we obtain same equations of motion for the gauge invariant quantities.

It has been shown that the added variables and their corresponding momenta can be eliminated at the end of the gauging process, using their constrained equation, if they are second class in comparison to other constraints [34].

Thus, the Lagrangian (2) is modified to

$$L^{(0)} = m\frac{\dot{x}.\dot{n}}{\sqrt{\dot{n}^2}} - \xi\cdot\dot{x} + \frac{1}{2}\dot{\xi}^2. \tag{19}$$

3.1. Symplectic Formalism of Modified Anyonic Model. In the previous part, we added some gauge symmetry to the anyonic model via (18). Here, we start the symplectic procedure to increase the gauge symmetries of the modified anyonic model [35]. A good review for the symplectic formalism can be found in [36].

Corresponding momenta, which are calculated before as relation (3), are now changed to the following relations, having three variables instead:

$$\begin{aligned} p_\mu &= \frac{m\dot{n}_\mu}{\sqrt{\dot{n}^2}} - \xi_\mu, \\ \wp_\mu &= \frac{m}{\sqrt{\dot{n}^2}}\left(\dot{x}_\mu - \frac{\dot{x}.\dot{n}}{\dot{n}^2}\dot{n}_\mu\right), \\ \pi_\mu &= \dot{\xi}_\mu. \end{aligned} \tag{20}$$

Calculating the canonical Hamiltonian, we have

$$H_c = \dot{x}\cdot p + \dot{n}\cdot\wp + \dot{\xi}\cdot\pi - L^{(0)} = \frac{1}{2}\pi^2 = V^{(0)}. \tag{21}$$

Apparently, this system has the following primary constraints:

$$\phi_0 = n^2 + 1, \tag{22}$$

$$\phi_1 = \Pi^2 - m^2, \tag{23}$$

$$\phi_2 = n\cdot\Pi, \tag{24}$$

$$\phi_3 = \wp\cdot\Pi, \tag{25}$$

where modified momenta is defined by $\Pi_\mu = p_\mu + \xi_\mu$. We observe that except for the constraint (5), all constraints in (6)–(8) are modified. This shows that (5) will remain intact during the embedding procedure, and thus, it can be ignored till the end of the process.

Now, by introducing constraints ϕ_i into the canonical sector of the first-order Lagrangian $L^{(0)}$, by means of the time derivative of Lagrange multipliers $\dot{\lambda}_i$, we get the first-iterative Lagrangian $L^{(1)}$ as

$$L^{(1)} = \dot{x}\cdot p + \dot{n}\cdot\wp + \dot{\xi}\cdot\pi - \sum_{i=1}^{3}\dot{\lambda}_i\phi^i - V^{(0)}, \tag{26}$$

where $V^{(0)}$ is obtained in (21).

Applying symplectic formalism, one can obtain the existing secondary constraint of the model as

$$\phi_4 = 4\pi \cdot \Pi. \tag{27}$$

It has been shown that one can construct the gauged Lagrangian by enlarging the phase space and adding a Wess-Zumino (WZ) term to the first-order Lagrangian as

$$\tilde{L}^{(1)} = L^{(1)} + L_{\mathrm{WZ}}, \tag{28}$$

where L_{WZ} is a function depending on the original coordinates and WZ variable, defining with the help of two generators $G^{(1)}$ and $G^{(2)}$ [27, 34]:

$$L_{\mathrm{WZ}} = G^{(1)} + G^{(2)}. \tag{29}$$

Checking the Poisson brackets of the constraints (23), (24), (25), and (27), we see that ϕ_1 is first class with respect to the others. So, the generators $G^{(1)}$ and $G^{(2)}$ are defined by following relation (38)

$$\begin{aligned} G^{(1)} &= \theta\phi_4, \\ G^{(2)} &= -\theta^2\{\phi_4, \phi_1\}. \end{aligned} \tag{30}$$

In the above equations, θ is the WZ variable, and its conjugate momentum, p_θ, which will not be appear in the gauged model, is a first-class constraint. Thus, it is the sign of the presence of the gauge symmetry in the obtained model.

Bringing this result into the first-order Lagrangian, we obtain the gauged Lagrangian as

$$\tilde{L}^{(1)} = \dot{x} \cdot p + \dot{n} \cdot \wp + \dot{\xi} \cdot \pi + \sum_{i=1}^{3} \dot{\lambda}_i \phi^i - \frac{1}{2}\pi^2 + 4\pi\Pi\theta - 8\Pi^2\theta^2, \tag{31}$$

and the embedded canonical Hamiltonian is read off as

$$\tilde{H}_c = H_c - G^{(1)} - G^{(2)}. \tag{32}$$

4. Poisson Structure of the Gauged Model

Now, we calculate all constraints' corresponding momenta, using $p_{\lambda i} = \partial\tilde{L}^{(0)}/\partial\dot{\lambda}^i$, with $\Phi_i = p_{\lambda i}$ $(i = 1, 2, 3)$ and $\Phi_4 = p_\theta$, and check out the consistency conditions

$$0 = \dot{\Phi}_i = \{\Phi_i, \tilde{H}_T\}, \tag{33}$$

where

$$\tilde{H}_T = \tilde{H}_c + \lambda_i \Phi^i. \tag{34}$$

Due to the fact that $\{\Phi_i, \Phi_j\} = 0$, secondary constraints can be obtained by calculating $\Psi_i = \{\Phi_i, \tilde{H}_c\}$. Thus,

$$\begin{aligned} \Psi_1 &= \Pi^2 - m^2, \\ \Psi_2 &= n \cdot \Pi, \\ \Psi_3 &= \wp \cdot \Pi, \\ \Psi_4 &= \pi \cdot \Pi + 4m^2\theta. \end{aligned} \tag{35}$$

Now, calculating the consistency condition for Ψ_is, we obtain the other part of the constraint chain structure.

$$\begin{aligned} \Lambda_1 &= m^2\theta, \\ \Lambda_2 &= n \cdot \pi - \lambda_3 m^2, \\ \Lambda_3 &= \wp \cdot \pi + \lambda_2 m^2. \end{aligned} \tag{36}$$

At this level, the consistency will be terminated. It is about time we should consider the primary constraint (5), which is left over during the symplectic procedure. From its consistency condition, we have

$$\Psi_0 = \lambda_3 \Lambda_2. \tag{37}$$

Finally, the constraint chain structure of the model is

$$\phi_0 \longrightarrow \Psi_0 \longrightarrow \lambda_3 \Lambda_2, \tag{38}$$

$$\Phi_1 \longrightarrow \Psi_1 \longrightarrow \Lambda_1, \tag{39}$$

$$\Phi_2 \longrightarrow \Psi_2 \longrightarrow \Lambda_2, \tag{40}$$

$$\Phi_3 \longrightarrow \Psi_3 \longrightarrow \Lambda_3, \tag{41}$$

$$\Phi_4 \longrightarrow \Psi_4. \tag{42}$$

As we see, we encountered a nondesirable bifurcation in the first and third lines of the chain structure. To overcome such a problem, we can propose two solutions.

First, If we try to eliminate λ_3, i.e., we get $\lambda_3 = 0$, our chains will be as follows:

$$\begin{aligned} &\phi_0 \longrightarrow \Psi_0 \longrightarrow \lambda_3 \quad \text{determines multiplier}, \\ &\Phi_1 \longrightarrow \Psi_1 \longrightarrow \Lambda_1, \\ &\Phi_2 \longrightarrow \Psi_2 \longrightarrow \Lambda_2 \quad \text{incontradiction to} \quad \lambda_3 = 0, \\ &\Phi_3 \longrightarrow \Psi_3 \longrightarrow \Lambda_3, \\ &\Phi_4 \longrightarrow \Psi_4. \end{aligned} \tag{43}$$

It is apparent that this contradicts our assumption.

Second and as the most proper approach, if we take $\Psi_0 = \lambda_3 = 0$, then the canonical couples $(\Phi_3 = p_{\lambda 3}, \lambda_3)$ are constraints themselves, and one can throw them away without considering them, when we are calculating Dirac brackets. We are allowed to do this elimination due to the fact that constraints which determine coefficients are second-class constraints, while others are first-class ones [33]. Therefore, the chain, $\Phi_3 \longrightarrow \Psi_3 \longrightarrow \Lambda_3$, will have never been taken into the account. Thus, in (38), we will have first, second, third, and fifth lines, only.

According to this consideration, we would have more first-class constraints, because the Poisson brackets depending on the mentioned chain structure are removed easily.

Rewriting the changed constraint according to our assumption, we have

$$\Lambda_2 = n \cdot \pi. \tag{44}$$

After extracting all possible constraints, using their simplified forms, we obtain only three first-class constraints as, ϕ_0, Φ_1, and Φ_2, and six second-class constraints in the model as Φ_4, Ψ_1, Ψ_2, Ψ_4, Λ_1, and Λ_2. This classification can be done with the help of the algorithm introduced in [33]. So, the corresponding Poisson bracket matrix of these second-class constraints will be

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & -16m^2 & -m^2 & 0 \\ 0 & 0 & 0 & 2m^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 16m^2 & -2m^2 & 0 & 0 & 0 & 0 \\ m^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

We will obtain the following Dirac brackets, which indicate the Poisson structure of the modified anyonic particle.

$$\left\{x_\mu, x_\nu\right\}^* = \frac{1}{m^2}\left(\pi_\mu \Pi_\nu - \pi_\nu \Pi_\mu\right), \tag{46}$$

$$\left\{x_\mu, \xi_\nu\right\}^* = n_\mu n_\nu - \frac{1}{m^2}\Pi_\mu \Pi_\nu, \tag{47}$$

$$\left\{x_\mu, p_\nu\right\}^* = \delta_{\mu\nu}, \tag{48}$$

$$\left\{x_\mu, \pi_\mu\right\}^* = \frac{-1}{m^2}\left(\pi_\mu \Pi_\nu - \pi_\nu \Pi_\mu\right), \tag{49}$$

$$\left\{x_\mu, \wp_\nu\right\}^* = -n_\mu \pi_\nu, \tag{50}$$

$$\left\{\xi_\mu, \pi_\nu\right\}^* = \delta_{\mu\nu} + n_\mu n_\nu - \frac{1}{m^2}\Pi_\mu \Pi_\nu, \tag{51}$$

$$\left\{\xi_\mu, \wp_\nu\right\}^* = n_\mu \Pi_\nu, \tag{52}$$

$$\left\{n_\mu, \wp_\nu\right\}^* = \delta_{\mu\nu}, \tag{53}$$

$$\{\theta, p_\theta\}^* = 0, \tag{54}$$

$$\left\{\pi_\mu, \pi_\nu\right\}^* = \frac{1}{m^2}\left(\pi_\mu \Pi_\nu - \pi_\nu \Pi_\mu\right), \tag{55}$$

$$\left\{\pi_\mu, \wp_\nu\right\}^* = n_\mu \pi_\nu, \tag{56}$$

$$\left\{\wp_\mu, \wp_\nu\right\}^* = -\left(\pi_\mu \Pi_\nu - \pi_\nu \Pi_\mu\right). \tag{57}$$

One can note that the cancellation of $\{\theta, p_\theta\}^*$ shows that θ is a gauge degree of freedom. Comparing (46) and (10), we see that the modified anyonic particle remained noncommutative but with a more extended phase space [30].

5. Physical Degrees of Freedom

The number of physical degrees of freedom can be obtained with the help of following relation:

$$\mathcal{N} = \#\left(\mathcal{Q}_i - \mathcal{FC}_i - \frac{1}{2}\mathcal{SC}_i\right), \tag{58}$$

where $\mathcal{Q}_i$ is the number of coordinates and $\mathcal{FC}_i$ and $\mathcal{SC}_i$ are the numbers of first-class and second-class constraints, respectively [37].

In a $(2+1)$ dimensional spacetime, the number of physical degrees of freedom of the original model is

$$\mathcal{N}_{\text{Original}} = 6 - 2 - 1 = 3, \tag{59}$$

while, for the modified model, it will be obtained as

$$\mathcal{N}_{\text{Modified}} = 12 - 3 - 3 = 6. \tag{60}$$

Apparently, the gauged model has three extra physical degrees of freedom in comparison with the original model. Generally, if we have all constraints solved in the model, the number of degrees of freedom does not change after the gauging process. But here, since we keep the auxiliary coordinates, we have more degrees of freedom in the gauged model, which can be interpreted as the interaction of the particle with the electromagnetic (gauge) field [1, 38–41].

6. Gauge Transformations' Generators

The generators of infinitesimal gauge transformations can be obtained with the help of Poisson brackets of the first-class constraints and the phase-space coordinates of Lagrangian (31), i.e., χ_α, via the following relation [37, 42]:

$$\delta\chi_\alpha^{(1)} = \left\{\chi_\alpha^{(1)}, \phi_j\right\}\varepsilon^j, \tag{61}$$

where ε_i are infinitesimal time-dependent parameters and ϕ_i are first-class constraints. So the infinitesimal gauge transformations of the modified anyonic particle model which determine its gauge symmetries are

$$\delta x_\mu = 0, \quad \delta p_\mu = 0, \tag{62}$$

$$\delta n_\mu = 0, \quad \delta\wp_\mu = -2n_\mu\varepsilon_0, \tag{63}$$

$$\delta\xi_\mu = 0, \quad \delta\pi_\mu = 0, \tag{64}$$

$$\delta\lambda_1 = \varepsilon_1, \tag{65}$$

$$\delta\lambda_2 = \varepsilon_2, \tag{66}$$

$$\delta\theta = 0. \tag{67}$$

Apparently, Lagrangian (31) and the corresponding Hamiltonian are invariant under these transformations.

7. Solving Auxiliary Coordinate and Its Effect on the Spin of Anyons

As we mentioned in previous parts, to gauge the Hamiltonian of a relativistic free any-spin particle, which is called an anyon so far, with symplectic embedding formalism, we extend the corresponding phase space with an auxiliary gauging variable, which finally resulted to three levels of constraints.

In this section, we intend to find a proper solution as an ungauge fixing for second-class constraints, to obtain a solved chain of constraints for the structure obtained in Section IV. To show the distinction between the solved constraints and intact ones, we put a prime sign over the name of all constraints, to notify that those constraints are simplified (imaged) on their constraint surface.

The first level of these three level constraints, indicating that the gauged model exists, is

$$\phi'_0 = n^2 + 1, \ \Phi'_1 = p_{\lambda 1}, \ \Phi'_2 = p_{\lambda 2}. \tag{68}$$

Among these constraints, ϕ'_0 is a first-class one, inherited unchanged from the former steps, describing anyons. The two others indicate that the gauging procedure has two trivial constraints, as the corresponding momenta of coordinates added through the phase-space extension.

Also, the gauging procedure changes the symplectic structure of the phase space which is shown by the following second-class constraints:

$$\begin{aligned} \Phi'_4 &= p_\theta, \\ \Psi'_1 &= \Pi^2 - m^2, \\ \Psi'_2 &= n \cdot \Pi, \\ \Psi'_4 &= \pi \cdot \Pi, \\ \Lambda'_1 &= \theta, \\ \Lambda'_2 &= n \cdot \pi. \end{aligned} \tag{69}$$

Then, the Poisson brackets of the second-class constraints include the terms containing only the gauging Chern-Simons variables, i.e., ξ_μ, and its corresponding momentum π_μ, not the symplectic variables of the space extension. Those variables are transplanted to the third category of constraints. Actually, they are first- or second-class constraints, where their existence does not affect the way we determine the phase space of the model. These constraints are $\Phi'_3, \Psi'_3, \Lambda'_3, \lambda_3$, and $p_{\lambda 3}$.

As the first result, conveyed from the nontrivial Poisson structure, the algebra of first-class constraints as the generators of gauge transformations of the gauged anyonic model is Abelian, although this property does not affect the algebra of the spins of the further quantum models.

Using the nonsymplectic Poisson structure and constraint equations (not equations of motion), one can find a dynamical solution for gauging Chern-Simons auxiliary coordinates (ξ_μ, π_μ) to remove them. This procedure adds a potential term to the relativistic anyonic model by affecting the kinematic part and changes the spin operator of the model. As we previously mentioned, adding the auxiliary coordinates has the advantage of making a complete second-class system, which can be gauged later more easily. Afterwards, these auxiliary coordinates should be removed. This solution should be consistent with the obtained symplectic structure.

Referring to the obtained Dirac brackets (46) and considering the second-class constraints and symplectic structure of the model, one can infer that the best choice is to get $\pi_\mu = 0$, which is a light-like condition for the auxiliary momentum $\pi^2 = 0$. This choice removes all terms containing π_μ, including the constraints Ψ'_4 and Λ'_2. The final symplectic structure is then obtained as

$$\{x_\mu, x_\nu\}^* = 0, \tag{70}$$

$$\{x_\mu, \xi_\nu\}^* = -n_\mu n_\nu - \frac{1}{m^2}\Pi_\mu \Pi_\nu, \tag{71}$$

$$\{x_\mu, p_\nu\}^* = \delta_{\mu\nu}, \tag{72}$$

$$\{\wp_\mu, \wp_\nu\}^* = 0, \tag{73}$$

$$\{x_\mu, \wp_\nu\}^* = 0, \tag{74}$$

$$\{n_\mu, \wp_\nu\}^* = \delta_{\mu\nu}, \tag{75}$$

$$\{\xi_\mu, \wp_\nu\}^* = n_\mu \Pi_\nu. \tag{76}$$

On the other hand, it seems that we are free to select any solution for ξ_μ, but preferably, we solve it with the constraints Ψ'_1 and Ψ'_2. In $(1+1)$ dimensions, these two relations completely solve $\xi = (\xi_0, \xi_1)$, but in $(2+1)$ dimensions, a free component will remain.

Anyway, as an important result, we see that the solution of ξ is a function of p as the anyon momentum, which means that ξ is related to the energy of the anyon. This important result will be studied from another point of view.

In famous previous anyonic works [43, 44], the nonphysical degree of freedom n_μ and its corresponding momentum $\wp_\mu$ are mostly used to make the spin operator of the particle, since they are not related to a physical event (recalling the constraint ϕ'_0). If we look at the constraint Ψ'_2 and the null constraint Ψ'_4, we see that ξ, which is regarded as a coordinate (not the coordinate of a particle with mass m, rather a massless particle $\pi^2 = 0$), is perpendicular to n, and therefore, it can contribute to the spin of the particle.

Thus, the anyonic spin relation can be generalized as

$$S_\mu = \varepsilon_{\mu\nu\sigma}\left(n^\nu \wp^\sigma + f(\xi, n, \wp) E^\nu \xi^\mu(p)\right), \tag{77}$$

where E is the electric part of the electromagnetic field, obtained from the gauge potential ξ. This relation indicates that the spin of the anyon is related to energy via $\xi_\mu(p)$ [45].

Another point is that, regarding (70), the scalar function f is determined in such a way where S_μ is an intrinsic spin, i.e., it satisfies the intrinsic spin algebra, and is separated from the classical angular momentum (see the second Dirac brackets of (70)). This spin operator contains a space-like vector $n^2 = -1$, in addition to a light-like vector $\pi^2 = 0$, and its corresponding coordinate, ξ_μ.

7.1. Some Hints over Relation (77). As we know, the light-like condition of a relativistic particle determines the thin border of the physical world (including any vector with light-like condition) and the nonphysical world (with space-like condition). Since quantum fluctuations may change the time-like nature of the vector π_μ to the light-like one $\pi_\mu^2 = 0$ and consequently changes the nonphysical vector n_μ to a light-like vector, then we can read off the following correction for the spin of the relativistic particle:

$$S_{\mu\nu}^{\text{new}} = S_{\mu\nu}^{\text{old}} + f(\xi)\left(n_\mu \xi_\nu - n_\nu \xi_\mu\right). \tag{78}$$

Here, n_μ and ξ_μ have the length and momentum (or length^{-1}) dimensions, respectively. The dimensionless function f adds the correction term of the spin via the angular momentum-like term $(n_\mu \xi_\nu - n_\nu \xi_\mu)$. Giving the length dimension to n_μ can be simply done by reintroducing the space-like condition $n^2 + d^2 = 0$, but one should notify that while n_μ is nonphysical, d_μ remains a nonphysical observable.

For instance, in the fundamental theory of gravity, d_μ is regarded as the Planck length, and in an effective anyonic theory in condensed matter physics, it can be chosen as a very small length, which emerges in a quantum phase transition (QPT) temperature of an order of nK and produces the temperature as

$$d = \frac{1}{K_B T_{\text{QPT}}}. \tag{79}$$

Now, let us imagine that the model experiences a small quantum fluctuation such as the creation of a particle (in the context of high energy physics), a severe resonance, or even an excitation (in the framework of condensed matter physics). This event happens during the interaction of the model and the corresponding background via the energy exchange. The Hamiltonian of a free anyon is written as

$$H = \frac{1}{2}\pi^2. \tag{80}$$

During this interaction, the energy function of the system and quantum background in the transition point, i.e., QPT temperature, is equal to

$$H = \frac{1}{2}\pi^2 - \frac{1}{2}d^2 p^2. \tag{81}$$

But there would be a situation where $\pi_\mu \sim p_\mu d$. This situation, which is called a resonance situation, puts the system in a dissipative phase, where the energy is leaked from the pseudophysical part to the pseudononphysical one. It is interesting to notice that the resonance width, i.e., the excited mode or the pseudoparticle lifetime which gives the energy uncertainty in the event, is related to the emergence of n in the model. Other properties of the resonance point can be obtained via the study of the kinematics and dynamics of the model.

8. Discussion

In relativistic classical mechanics point of view, the basic Lagrangian of the anyons are constructed with the help of some nonphysical coordinates, such as n_μ, with the space-like condition $n^2 + 1 = 0$. The corresponding angular momenta of these coordinates make an any-spin particle in the quantization process. Classically, relating physical degrees of freedom of a particle to its quantum spin is an unsolved problem.

Here, we address this problem by making a gauged model via the symplectic formalism of constrained systems. Since the primary model is a mixed model of both first- and second-class constraints, the embedding gauging procedure is unsuccessful. Thus, to perform a gauging process, we need a pure second-class constraint, which is obtained by adding an auxiliary couple of conjugate variables, changing the model in a way we desire. Since the constraint $n^2 + 1 = 0$ remains intact during the gauging process, one cannot couple the auxiliary field with the help of n_μ, and instead, this is done with the help of a new coordinate ξ_μ and its corresponding momentum π_μ. Keeping these variables unsolved to the end of the gauging process, we gave them the desirable physical properties.

By looking to the second level of the constraints of the gauged model, we see that they mostly contain the auxiliary couple (ξ_μ, π_μ). Although this couple is not a plainly second-class constraint, they are related to the symplectic part of the gauged model and the corresponding second-class constraints. They also add a solved gauge field to the anyonic particle, where its solutions are obtained with respect to the primary coordinates (ξ_μ, π_μ). This solved gauge field, which is minimally coupled to the phase-space coordinates, can be explained as the electromagnetic-like interaction with the anyon. The corresponding constraint of this auxiliary field, $\pi^2 = 0$, shows that the field carrying the electromagnetic-like interaction for the anyon is massless, like the photon. One should note that the corresponding coupling is only done by converting the SCCs to the FCCs.

In the context of the theory of a constrained system, one can consider a part of second-class constraints as first-class ones accompanying their fixing conditions. Thus, by dividing the second-class constraints into two parts via a suitable approach, one can keep a part of them and impose another part as the solutions, to obtain a first-class (say gauged) system [46]. Using this method, we witness that the solution $\pi_\mu = 0$ solves two second-class constraints Ψ'_4 and Λ'_2 and leaves two first-class ones Ψ'_1 and Ψ'_2. This solution also adds the coordinate ξ_μ to the Hamiltonian in the same way where a gauge vector potential is added via minimal coupling.

Hence, we added two types of the gauge symmetries to the free relativistic particle model, where the spin of the anyon is obtained from. The first is the global gauge symmetry which is written via the variation terms (62) and obtained via the symplectic embedding procedure. This symmetry remains intact, even after the gauge fixing of constraints Ψ'_1 and Ψ'_2 is applied.

The second type is the local gauge symmetry, which is obtained via ungauge fixing of the second-class constraints of the model. Here, the locality of gauge symmetry refers to the momentum part of the phase space (not the configuration space), which leads to the dependency of the spin of the anyon to its momentum.

Also, as we saw, after gauging the anyon's Hamiltonian, we derived an interacting model, related to the correction terms which appeared in the gauged model, which can be considered as the interaction of anyons and the electromagnetic (gauge) field. Extracting and comparing these terms with others [1, 38, 40] may provide interesting results.

As another result, one can see the ungauge-fixing process with the second-class constraints Ψ'_4 and Λ'_2, to write the embedded Hamiltonian in a more standard form. These two primarily canonical constraints are changed to noncanonical forms after the embedding procedure and make the symplectic structure, i.e., $\{\theta, p_\theta\}^* = 0$. This means that they correspond to a component of a canonical couple, e.g., proposing the following solution for them

$$\theta = p_\kappa, \quad p_\theta = p_\kappa, \quad \{\kappa, p_\kappa\} = 1. \tag{82}$$

This choice results in the terms related to θ, which is added to the Hamiltonian via the embedding procedure, and appeared as the form of the nonrelativistic kinetic term of the new gauge degrees of freedom. Moreover, the coordinate of this degree of freedom is a cyclic coordinate.

In the end, as we mentioned previously, by selecting a specific solution for ξ_μ in $(1+1)$ dimensions, one can completely solve ξ_μ components. This is an interesting result and can open a field of study in the context of conformal field theory (CFT) in $(1+1)$ dimensions, since it has been shown that anyonic models, especially those which are studied in the context of the quantum Hall effect, can be represented as conformal blocks of a particular two-dimensional CFT [47–51]. Moreover, due to the analogy existing between the conformal blocks of CFT and the anyon wave functions, there is a relation between the two-dimensional CFT and the anyon theory, in the context of the Chern-Simons field theory [51]. This relation is clearer between the $(2+1)$ dimensional Chern-Simons theory and the two-dimensional CFT [52, 53].

Hence, for further works, it would be interesting to study the relation of the Virasoro algebra in the context of anyon theories and to investigate the nonlinear extension of $\mathscr{W}$ algebras.

Data Availability

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

Disclosure

The research was performed as part of the employment of the authors.

Conflicts of Interest

The authors also declare that there is no conflict of interests regarding the publication of this paper.

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