Asian Research Journal of Mathematics

17(2): 72-90, 2021; Article no.ARJOM.*67314*
ISSN: 2456-477X

Analyzing a New Third Order Iterative Method for Solving Nonlinear Problems

M. S. Ndayawo[1*] and B. Sani[2]

[1]Department of Mathematics and Statistics, Kaduna Polytechnic, Kaduna, Nigeria.
[2]Department of Mathematics, Ahmadu Bello University Zaria, Nigeria.

Authors' contributions

This work was carried out in collaboration between both authors. Author MSN designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author BS managed the analyses of the study managed the literature searches. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2021/v17i230275
Editor(s):
(1) Dr. Ruben Dario Ortiz Ortiz, Universidad Michoacana de San Nicolas de Hidalgo, Mexico.
Reviewers:
(1) Yazhneg Dang, University of Shanghai for Science and Technology, China.
(2) Sabah Hassan Malih, University of Baghdad, Iraq.
(3) Yanlai Song, Zhongyuan University of Technology, China.
Complete Peer review History: http://www.sdiarticle4.com/review-history/67314

Original Research Article

Received 09 February 2021
Accepted 17 April 2021
Published 23 April 2021

Abstract

In this paper, we propose and analyse a new iterative method for solving nonlinear equations. The method is constructed by applying Adomian method to Taylor's series expansion. Using one-way analysis of variance (ANOVA), the method is being compared with other existing methods in terms of the number of iterations and solution to convergence between the individual methods used. Numerical examples are used in the comparison to justify the efficiency of the new iterative method.

Keywords: Iterative methods; ANOVA; nonlinear equation; order of convergence.

1 Introduction

Finding zeros or roots of one variable nonlinear equations, $f(x)=0$ efficiently is a significant discussion in numerical analysis and has broad span of applications in all areas of science and engineering. Choosing a single method from the numerous different methods developed by researchers to get analytic solutions of such equations is strenuous, as a result of which various researchers are continuously coming up with new

**Corresponding author: Email: muhdshakurn@gmail.com;*

numerical methods for solving nonlinear equations. Some of these methods were developed using Newton-Raphson method (NR), Taylor's method and methods [1-3] to mention few. Presently, iterative methods are being developed by combining two or more existing methods which results in improving the convergence rate, better accuracy as well as iteration perspective for solving nonlinear equations. Nasr [4] proposed two iterative methods for solving nonlinear algebraic equations by Using Least Square Method" which was developed by combining Rafiullah and Jabeens [5] method and the least square method. The two methods have eighth and sixteenth-order rate of convergence. Napassanan and Montri [6], presented a new higher order iterative method for solving nonlinear equations. The method is based on both Halley's method see Albeanu [7] and the predictor-corrector technique. The convergence analysis shows that the method is of seventh order. Saqibi and Iqbal [8] presented and analysed two new methods which have fourth and fifth order convergence. The methods were developed by rewriting the nonlinear equations as a coupled system and applying modified decomposition technique. The efficiency and performance of the methods were compared with those of some existing methods. Alyauma [9] developed a two- step iterative method for solving a nonlinear equation, which is derivative free. The method has a convergence of order four and the method requires three evaluations of functions per iteration.

In this paper, we introduce a new iterative method for solving nonlinear equations without second derivative. The method has a third order rate of convergence and it is developed by applying Adomian decomposition method to Taylor's series expansion around x of higher order. In the method, the assumption that $\frac{f''(\gamma)}{f'(\gamma)} \approx m$, where *m* is a real number is used. This is done in order to eliminate the evaluation of second derivative in the new scheme. The presence of second derivative in an iterative scheme is usually a drawback. The new proposed scheme is then compared with other existing iterative schemes. The convergence of the proposed method is faster in many cases from the tested numerical examples.

2 Description of Adomian Decomposition method

Adomian decomposition method is applied to solve problems in mathematics, engineering and other related fields. This method does not require any assumption or linearization to solve any given problem. The idea is as follows:

Consider the equation:

$$Fu = g(t,x),$$

where *F* is a differential operator involving linear and nonlinear terms. Rewriting the equation in operator form as

$$Lu + Ru + Nu = g \tag{1}$$

where *L* is the highest order derivative which is easily invertible, *R* is the remainder of the linear differential portion, and *N* is a nonlinear operator. Solving (1) and since L is invertible we get

$$u = L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu$$

Since *F* is taken to be a differential operator and *L* is linear, L^{-1} would represent integration and with the given initial or boundary conditions,

The solutions of (1) consist of approximate solutions as an infinite series

$$u(t)=\sum_{n=0}^{\infty}u_n(t)$$

Decomposing the nonlinear term into a series of Adomian polynomials, as

$$Nu=\sum_{n=0}^{\infty}A_n$$

Where A_n are called the Adomian polynomials depending on $u_0,u_1,\cdots,u_n$.

To determine the Adomian polynomials, a grouping parameter, λ is introduced. It should be noted that λ is not a "smallness parameter".

$$u(t)=\sum_{n=0}^{\infty}\lambda^n u_n \tag{2}$$

And

$$Nu(\lambda)=\sum_{n=0}^{\infty}\lambda^n A_n \tag{3}$$

Then

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0},n=0,1,2,\cdots$$

The first few polynomials are given as

$$A_0=N(x_0),$$
$$A_1=x_1N'(x_0),$$
$$A_2=x_2N'(x_0)+\frac{1}{2}x_1^2N''(x_0),$$
$$A_3=x_3N'(x_0)+x_1x_2+\frac{1}{3!}x_1^3N''(x_0),$$
$$\vdots$$

2.1 Formation of the new method

Consider the nonlinear equation,

$$f(x)=0 \tag{4}$$

If α is a root of (4), and γ is the initial guess sufficiently close to α, then (4) can be rewritten using Taylor's series, see Burden and Faires [10], so that

$$f(\gamma)+f'(\gamma)(x-\gamma)+f''(\gamma)\frac{(x-\gamma)^2}{2}+g(x)=0 \quad (5)$$

where $g(x)$ represents the truncated part of higher order from the third term.

Rearranging (5) we get the following equation,

$$g(x)=f(x)-f(\gamma)-f'(\gamma)(x-\gamma)-f''(\gamma)\frac{(x-\gamma)^2}{2}, \text{ for } f(x)=0 \quad (6)$$

And from (5) we obtain

$$f'(\gamma)(x-\gamma)=-f(\gamma)-f''(\gamma)\frac{(x-\gamma)^2}{2}-g(x)$$

$$x-\gamma=-\frac{f(\gamma)}{f'(\gamma)}-\frac{f''(\gamma)}{f'(\gamma)}\frac{(x-\gamma)^2}{2}-\frac{g(x)}{f'(\gamma)}$$

$$x=\gamma-\frac{f(\gamma)}{f'(\gamma)}-m\frac{(x-\gamma)^2}{2}-\frac{g(x)}{f'(\gamma)}, \quad (7)$$

$\{$*on the assumption that* $\frac{f''(\gamma)}{f'(\gamma)}\approx m,$ where m is a real number$\}$

$$\therefore x=c+N(x) \quad (8)$$

where $c=\gamma-\frac{f(\gamma)}{f'(\gamma)}$

and

where $N(x)=-m\frac{(x-\gamma)^2}{2}-\frac{g(x)}{f'(\gamma)}$ is a nonlinear function. (9)

Comparing this analogously with the Adomian decomposition solution series we obtain $x=\sum_{n=0}^{\infty}x_n$, where the nonlinear function is given as $N(x)=\sum_{n=0}^{\infty}A_n$

$$\Rightarrow x = \sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n \text{ from which } x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}.$$

If $x = x_0 = \gamma$ for initial guess,

then from (6),

$$g(x) = f(x) - f(\gamma) - f'(\gamma)(x-\gamma) - f''(\gamma)\frac{(x-\gamma)^2}{2}$$

$$\Rightarrow g(x) = f(x)$$

$$\therefore g(x_0) = f(x_0)$$

$$N(x_0) = -m\frac{(x_0-\gamma)^2}{2} - \frac{g(x_0)}{f'(\gamma)} = -m\frac{(x_0-\gamma)^2}{2} - \frac{f(x_0)}{f'(\gamma)} = A_0, \text{ since } A_0 = N(x_0).$$

from (6) and (9) we have

$$N(x) = -m\frac{(x-\gamma)^2}{2} - \frac{f(x)}{f'(\gamma)} + \frac{f(\gamma)}{f'(\gamma)} + \frac{f'(\gamma)}{f'(\gamma)}(x-\gamma) + m\frac{(x-\gamma)^2}{2} \tag{10}$$

Ignoring the fifth term of the R.H.S of equation (10) and differentiating, we get

$$N'(x) = -m(x-\gamma) - \frac{f'(x)}{f'(\gamma)} + 1 = -m\left(\gamma - \frac{f(\gamma)}{f'(\gamma)} - \gamma\right) - \frac{f'(x)}{f'(\gamma)} + 1, \text{ since } x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}$$

$$\therefore N'(x_0) = 1 + m\frac{f(\gamma)}{f'(\gamma)} - \frac{f'(x_0)}{f'(\gamma)}$$

$$A_1 = x_1 N'(x_0) = \frac{-f(x_0)}{f'(\gamma)}\left\{1 + m\frac{f(\gamma)}{f'(\gamma)} - \frac{f'(x_0)}{f'(\gamma)}\right\} = -\frac{f(x_0)}{f'(\gamma)} - m\frac{f(x_0)f(\gamma)}{f'^2(\gamma)} + \frac{f(x_0)f'(x_0)}{f'^2(\gamma)} = x_2$$

Since $x_1 = \dfrac{-f(x_0)}{f'(\gamma)}$ from Appendix III

The value of x is approximated by $X_m = x_0 + x_1 + \cdots + x_m = x_0 + A_0 + A_1 + \cdots + A_{m-1}$

where $\lim_{m\to\infty} X_m = x$.

For m=0

$$x = X_0 = x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For m=1

$$x = X_1 = x_0 + x_1 = c + A_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_{n+1}^*)}{f'(x_n)} \text{ where } x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}$$

For m=2

$$x = X_2 = x_0 + x_1 + x_2 = c + A_0 + A_1 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} - m\frac{f(x_0)f(\gamma)}{f'^2(\gamma)} + \frac{f(x_0)f'(x_0)}{f'^2(\gamma)}$$

$$= \gamma - \frac{f(\gamma)}{f'(\gamma)} - 2\frac{f(x_0)}{f'(\gamma)} - m\frac{f(x_0)f(\gamma)}{f'^2(\gamma)} + \frac{f(x_0)f'(x_0)}{f'^2(\gamma)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2\frac{f(x_{n+1}^*)}{f'(x_n)} - m\frac{f(x_{n+1}^*)f(x_n)}{f'^2(x_n)} + \frac{f(x_{n+1}^*)f'(x_{n+1}^*)}{f'^2(x_n)} \tag{11}$$

3 Convergence Analysis

Theorem 3.1; Let $\alpha \in I$ be a simple root of sufficiently differentiable function $f : I \to R$ for an open interval *I*. Then the New method (11), has a third order rate of convergence and satisfy the following error equation.

$$e_{n+1} = (c_2 - m)e_n^3 + 0(e_n^4)$$

Proof

Let α be a simple root of $f(x)$ i.e. $f(\alpha) = 0,\ f'(\alpha) \neq 0.$ Assume e_n to be the error at the *nth* iteration so that $e_n = x_n - \alpha,$ By Taylors series expansion

$$f(x_n) = f(\alpha + e_n)$$

$$f\left(x_n\right)=f\left(\alpha\right)+f'\left(\alpha\right)e_n+\frac{f''\left(\alpha\right)}{2!}e_n^2+\frac{f'''\left(\alpha\right)}{3!}e_n^3+\frac{f'^{v}\left(\alpha\right)}{4!}e_n^4+0\left(e_n^5\right)$$

$$=f'\left(\alpha\right)\left[e_n+\frac{f''\left(\alpha\right)}{2f'\left(\alpha\right)}e_n^2+\frac{f'''\left(\alpha\right)}{3!f'\left(\alpha\right)}e_n^3+\frac{f'^{v}\left(\alpha\right)}{4!f'\left(\alpha\right)}e_n^4+0\left(e_n^5\right)\right],\text{ since } f(\alpha)=0,$$

$$\therefore f\left(x_n\right)=f'\left(\alpha\right)\left[e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+0\left(e_n^5\right)\right], \tag{12}$$

$$f'\left(x_n\right)=f'\left(\alpha\right)\left[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+0\left(e_n^4\right)\right],$$

where $c_j=\frac{1}{\text{j}!}\frac{f^j\left(\alpha\right)}{f'\left(\alpha\right)},\ j=2,3,\cdots$

$$\text{i.e. } c_2=\frac{f^2\left(\alpha\right)}{2f'\left(\alpha\right)}\Rightarrow 2c_2=m\text{ ,since }\frac{f''\left(\alpha\right)}{f'\left(\alpha\right)}\approx m\text{ and } e_n=x_n-\alpha \tag{13}$$

$$\therefore f'\left(x_n\right)=f'\left(\alpha\right)\left[1+me_n+3c_3e_n^2+4c_4e_n^3+0\left(e_n^4\right)\right] \tag{14}$$

and

$$f'^2\left(x_n\right)=f'^2\left(\alpha\right)\left[1+2me_n+\left(6c_3+m^2\right)e_n^2+\left(6mc_3+8c_4\right)e_n^3+\left(0\right)e_n^4\right] \tag{15}$$

From equations (12) and (14) we have

$$\frac{f\left(x_n\right)}{f'\left(x_n\right)}=\left\{e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+0\left(e_n^5\right)\right\}\left\{1+me_n+3c_3e_n^2+4c_4e_n^3+0\left(e_n^4\right)\right\}^{-1}$$

$$=e_n+\left(c_2-m\right)e_n^2+\left(-mc_2-2c_3+m^2\right)e_n^3+0\left(e_n^4\right)$$

$$\therefore\frac{f\left(x_n\right)}{f'\left(x_n\right)}=e_n+\left(c_2-m\right)e_n^2+\left(-mc_2-2c_3+m^2\right)e_n^3+0\left(e_n^4\right)$$

but

$$x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)}=x_{n+1}^*=\alpha+e_n-\left[e_n+\left(c_2-m\right)e_n^2+\left(-mc_2-2c_3+m^2\right)e_n^3+0\left(e_n^4\right)\right] \tag{16}$$

and so

$$x_{n+1}^*=\alpha+\left(mc_2+2c_3-m^2\right)e_n^3-\left(c_2-m\right)e_n^2+0\left(e_n^4\right)$$

Thus the error for x_{n+1}^* is

$$\left(mc_2+2c_3-m^2\right)e_n^3-\left(c_2-m\right)e_n^2+0\left(e_n^4\right) \tag{17}$$

Using equation (12), we can obtain the value of $f\left(x_{n+1}^*\right)$ by substituting e_n in equation (17), and we get:

$$f\left(x_{n+1}^*\right)=f'(\alpha)\left\{-\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3+c_2\left[-\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3\right]^2+0\left(e_n^4\right)\right\}$$

$$\Rightarrow f\left(x_{n+1}^*\right)=f'(\alpha)\left\{-\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3+0\left(e_n^4\right)\right\} \tag{18}$$

Similarly, using equation (14), we can obtain the value of $f'\left(x_{n+1}^*\right)$, substituting e_n in equation (17). This gives:

$$f'\left(x_{n+1}^*\right)=f'(\alpha)\left\{1-m\left[\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3+0\left(e_n^4\right)\right]+3c_3\left[-\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3+0\left(e_n^4\right)\right]^2+\cdots\right\}$$

$$\Rightarrow f'\left(x_{n+1}^*\right)=f'(\alpha)\left\{1-m\left[\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3\right]+0\left(e_n^4\right)\right\}$$

$$\therefore f'\left(x_{n+1}^*\right)=f'(\alpha)\left\{1-\left[\left(mc_2-m^2\right)e_n^2+\left(m^2c_2+2mc_3-m^3\right)e_n^3\right]+0\left(e_n^4\right)\right\} \tag{19}$$

Using equations (14) and (18) we get

$$\frac{f\left(x_{n+1}^*\right)}{f'\left(x_n\right)}=\frac{-\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3+0\left(e_n^4\right)}{1+me_n+3c_3e_n^2+4c_4e_n^3+0\left(e_n^4\right)}$$

$$=\left(m(m)+2c_3-2m^2\right)e_n^3-\left(c_2-m\right)e_n^2+0\left(e_n^4\right)\ \{\text{since } 2c_2=m \ \ see\ (13)\}$$

$$=\left(2c_3-m^2\right)e_n^3-\left(c_2-m\right)e_n^2+0\left(e_n^4\right)$$

$$\Rightarrow 2\frac{f\left(x_{n+1}^*\right)}{f'\left(x_n\right)}=2\left\{\left(2c_3-m^2\right)e_n^3-\left(c_2-m\right)e_n^2+0\left(e_n^4\right)\right\}$$

$$=\left(4c_3-2m^2\right)e_n^3-\left(2c_2-2m\right)e_n^2+0\left(e_n^4\right)$$

$$=\left(4c_3-2m^2\right)e_n^3-\left(m-2m\right)e_n^2+0\left(e_n^4\right)$$

$$\therefore 2\frac{f\left(x_{n+1}^*\right)}{f'\left(x_n\right)}=\left(4c_3-2m^2\right)e_n^3+me_n^2+0\left(e_n^4\right) \tag{20}$$

Using equations (12), (15) and (18) we get

$$\frac{f\left(x_n\right)f\left(x_{n+1}^*\right)}{f'^2\left(x_n\right)}=\frac{\left\{e_n+c_2e_n^2+c_3e_n^3+0\left(e_n^4\right)\right\}\left\{-\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3+0\left(e_n^4\right)\right\}}{1+\left(6mc_3+8c_4\right)e_n^3+\left(6c_3+m^2\right)e_n^2+2me_n+\left(0\right)e_n^4}$$

$$\therefore m\frac{f\left(x_n\right)f\left(x_{n+1}^*\right)}{f'^2\left(x_n\right)}=\left(m^2-mc_2\right)e_n^3+\left(0\right)e_n^4 \qquad (21)$$

Again, using equations (18), (19) and (15) we get

$$\frac{f\left(x_{n+1}^*\right)f'\left(x_{n+1}^*\right)}{f'^2\left(x_n\right)}=$$

$$\frac{\left\{-\left(c_2-m\right)e_n^2+\left(mc_2+2c_3-m^2\right)e_n^3+0\left(e_n^4\right)\right\}\left\{1-\left[\left(mc_2-m^2\right)e_n^2+\left(m^2c_2+2mc_3-m^3\right)e_n^3+\left(0\right)e_n^4\right]\right\}}{\left[1+2me_n+\left(6c_3+m^2\right)e_n^2+\left(6mc_3+8c_4\right)e_n^3+\left(0\right)e_n^4\right]}$$

$$=\left(m-c_2\right)e_n^2+\left(mc_2+2c_3-2m^2\right)e_n^3+\left(0\right)e_n^4$$

$$\therefore\frac{f\left(x_{n+1}^*\right)f'\left(x_{n+1}^*\right)}{f'^2\left(x_n\right)}=\left(m-c_2\right)e_n^2+\left(mc_2+2c_3-2m^2\right)e_n^3+\left(0\right)e_n^4 \qquad (22)$$

Finally, using equations (16), (20), (21) and (22) in (11) we get

$$x_{n+1}=\alpha+\left(mc_2+2c_3-m^2\right)e_n^3-\left(c_2-m\right)e_n^2-\left(4c_3-2m^2\right)e_n^3-me_n^2-\left(m^2-mc_2\right)e_n^3+\left(m-c_2\right)e_n^2$$
$$+\left(mc_2+2c_3-2m^2\right)e_n^3+\left(0\right)e_n^4$$

$$=\alpha+\left(3mc_2-2m^2\right)e_n^3+\left(-2c_2+m\right)e_n^2+0\left(e_n^4\right)$$

$$\Rightarrow x_{n+1}-\alpha=\left(mc_2-m^2\right)e_n^3+0\left(e_n^4\right)$$

$$\therefore e_{n+1}=\left(mc_2-m^2\right)e_n^3+0\left(e_n^4\right) \qquad (23)$$

equation (23) proves that the New method (11) has a third order rate of convergence.

To demonstrate the behavior and effectiveness of the new method, a comparison between the new method and other good and competent methods was conducted using fifty distinct problems.

The contrast is based on the number of iterations that every method takes before the solution is reached. The methods used in the comparison are:

i. The New method
ii. Noor's method [11]

iii. Newton Raphson's method
iv. Chun's method [12]
v. Basto et al's method [13]
vi. Ndayawo and Sani [14]

The complete fifty problems included in the analysis are in Appendix I, while the details of the computations are in Appendix II.

3.1 Computation and analysis of results

Computation of $\dfrac{f''(x)}{f'(x)} \approx m \in \mathbb{R}$

To use the new iterative method, we need to get the value of *m*. We approximate *m* from $\dfrac{f''(x)}{f'(x)}$ where *x* is the initial point. For instance, to obtain the value of *m* for the first problem in Appendix I,

i.e, $f(x) = \sin^2 x - x^2 + 1$, at $x = 1.3$

we obtain $m \approx \dfrac{f''(x)}{f'(x)} = \dfrac{2\cos^2 x - 2\sin^2 x - 2}{2\sin x \cos x - 2x} = 1.781616671$ at $x = 1.3$ where *x* is in radians.

which is substituted into equation (11) to get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2\frac{f(x^*_{n+1})}{f'(x_n)} - m\frac{f(x^*_{n+1})f(x_n)}{f'^2(x_n)} + \frac{f(x^*_{n+1})f'(x^*_{n+1})}{f'^2(x_n)}$$

Statistical analysis was carried out from the numerical data to ascertain the findings. A one-way ANOVA test was conducted and the following results were obtained taking into account, 95 percent confidence interval. As a result, we find a significant difference in the number of iterations between the methods analysed, since $P = 0.00 < 0.05$, see Table 1. Also from Table 2, since $P = 0.998 > 0.05$, for the solutions, we conclude that there is no significant difference in the average solutions obtained by the methods used.

Table 1. One-Way ANOVA results for number of iterations to convergence

	Sum of Squares	Df	Mean Square	F	Sig.
Between Groups	159.532	5	31.906	17.910	0.000
Within Groups	479.210	269	1.781		
Total	638.742	274			

Table 2. One-Way ANOVA results for solution values at convergence

	Sum of Squares	Df	Mean Square	F	Sig.
Between Groups	2.340	5	0.468	0.056	0.998
Within Groups	2229.456	269	8.288		
Total	2231.796	274			

The complete results of the number of iterations obtained for all the tested methods across all the 50 problems are in Appendix II. To analyse the difference in the number of iterations, we used the post Hoc

Test (Duncan Multiple range) and from the results obtained, which are in Table 3, we see that Chun's method, Basto et al's method and the New method have the least number of iterations all in the first homogeneous subset and Newton Raphson method in the second homogeneous subset. In the third homogeneous subset we have Ndayawo and Sani's method and Noor's method, with higher number of iterations.

Table 3. Homogeneous subsets (Post Hoc Test) showing number of iterations to convergence in respect of the methods

Methods	N	Subset for alpha = 0.05		
		1	**2**	**3**
Chun	49	2.69		
Basto	45	2.84		
New Method	50	2.94		
Newton Raphson	46		3.72	
Ndayawo & Sani	45			4.29
Noor	40			4.75
Sig.		0.411	1.000	0.100

As can be seen from Table 4 below, the descriptive statistics, shows the mean, standard deviation and standard error of the number of iterations to convergence in respect of each of the methods.

Basto et al's method has the least standard deviation of 0.796 and standard error of 0.119, but five of the tested problems diverge, which is followed by Newton-Raphson's method with a standard deviation of 1.167 and standard error of 0.172, and four of tested problems did not converge. Next with a standard deviation of 1.185 and a standard error of 0.168 is the New method and none of the tested problems diverge. Then Chun's method with a standard deviation of 1.294 and standard error of 0.185 and one of the tested problems diverges.

Table 4. Descriptive statistics for each method in respect of number of iterations to convergence

	No	Mean	Standard Deviation	Standard Error	95% Confidence Interval for Mean		Min	Max
					Lower Bound	**Upper Bound**		
Noor	40	4.75	1.645	0.260	4.22	5.28	2	9
Newton Raphson	46	3.72	1.167	0.172	3.37	4.06	1	7
Chun	49	2.69	1.294	0.185	2.32	3.07	1	8
Ndayawo & Sani	45	4.29	1.753	0.261	3.76	4.82	2	11
Basto	45	2.84	0.796	0.119	2.61	3.08	2	5
New Method	50	2.94	1.185	0.168	2.60	3.28	1	6
Total	275	3.49	1.527	0.092	3.31	3.68	1	11

4 Conclusion

In this paper, we present a new iterative method for solving nonlinear equations. The new method is compared with four other existing good methods for the number of iterations to convergence. There are fifty problems used in the comparison which are given in Appendix I, while the complete results of the number of iterations in respect of each method are in Appendix II. From the analysis of variance results obtained, it shows that there is a significant difference between the number of iterations to convergence obtained from the methods used. The New method converges to all the fifty solutions, Chun's method converges to fortynine out of the fifty solutions while there is divergence in four of the solutions in the case Newton Raphson. For the Basto et al's method, five of the solutions of the test problems diverge. Lastly, for Noor's method, solutions of ten out of the fifty tested problems diverge.

As for the number of iterations, Chun's method has the smallest mean of 2.69 followed by Basto's method and New method with 2.84 and 2.94 respectively. In the case of the standard deviation, Basto's method has the best standard deviation of 0.796 followed by Newton Raphson's method and New method with standard deviations of 1.167 and 1.307 respectively. Thus the New method is comparatively good in all the phases being considered.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Adomian G. Nonlinear Stochastic Operator Equations, Academic Press Orlando, Florida; 1986.

[2] Daftardar-Gejji V, Jafari H. An iterative method for solving nonlinear functional equations. J Math Anal Appl. 2006;316:753–763.

[3] He JH. Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering 1999;178: 257–262.

[4] Nasr AI. Improvement of new eight and sixteenth order iterative methods for solving nonlinear algebraic equations by using least square method, International Journal of Scientific and Innovative Mathematical Research. 2018;6(10):23-27.

[5] Rafiullah M, Jabeens D. New eighth and sixteenth order iterative methods to solve nonlinear Equations. International Journal of Applied and Computational Mathematics. 2017;3:2467–2476.

[6] Napassanan S, Montri T. A note on three-step iterative method with seventh order of convergence for solving nonlinear equations. Thai Journal of Mathematics. 2016;14(3):565–573.

[7] Albeanu G. On the generalized Halley method for solving nonlinear equations, Romai Journal. 2008; 4(2):1–6.

[8] Saqib M, Iqbal M. Some multi-step iterative methods for solving nonlinear equations. Open Journal of Mathematical Sciences. 2017;1:25-33.

[9] Alyauma H. Solving a nonlinear equation using a new two-step derivative-free iterative methods. Applied and Computational Mathematics. 2017;6(6):238-242.

[10] Burden RL, Faires JD. Numerical Analysis, Ninth Edition Brooks/Cole, California; 2011.

[11] Noor MA. New family of iterative methods for nonlinear equations. Applied Mathematics and Computation. 2007; 190: 553-558.

[12] Chun C. Iterative methods improving Newton's method by the decomposition method, Computers and Mathematics with Applications. 2005;50:1559–1568.

[13] Basto M, Semiao V, Calheiros F. A new iterative method to compute nonlinear equations. Applied Mathematics and Computation. 2006;173:468–483.

[14] Ndayawo MS, Sani B. New iterative schemes for solving nonlinear equations, Journal of the Nigeria Association of Mathematical Physics. 2013;25(1):425-438.

[15] Chun C, Neta B. Some modification of Newton method by the method of undetermined coefficients, Journal of Computers and Mathematics with Applications. 2008;56:2528-2538.

[16] Kumar M, Singh AK, Srivastava A. Various Newton-type Iterative Methods for Solving Nonlinear Equations, Journal of the Egyptian Mathematical Society. 2013;21:34-339.

[17] Soleymani F. Novel computational iterative methods with optimal order for nonlinear equations, Hindawi Publishing Corporation Advances in Numerical Analysis. 2011 ID 2011; 270903.

Appendix I. Fifty problems used in the comparison

Fifty different Tested Problems

1. $\sin^2 x - x^2 + 1 = 0$

2. $x^2 - e^x - 3x + 2 = 0$

3. $\cos x - x = 0$

4. $(x-1)^3 - 1 = 0$

5. $x^3 - 10 = 0$

6. $xe^{x^2} - \sin^2 x + 3cosx - x = 0$

7. $xe^{x^2} - \sin^2 x + 3cosx + 5 = 0$

8. $e^{x+7x-30} - 1 = 0$

9. $e^{2x} + 2x + 0.1 = 0$

10. $2x^3 - x^2 - 7x + 6 = 0$

11 $x^3 - 9x + 1 = 0$

12. $x^3 - 3x - 1.06 = 0$

13. $x^3 - 6x + 4 = 0$

14. $2x - 3\sin x - 5 = 0$

15. $x^3 - 3x + 1 = 0$

16. $3x - \ln x - 16 = 0$

17. $\cos x - 2x + 3 = 0$

18. $x + \ln x - 2 = 0$

19. $x^3 - x - 0.1 = 0$

20. $x^4 - 12x + 7 = 0$

21. $x + \sin x + 0.5 = 0$

22. $x - 2 - e^{-x} = 0$

23. $x^3 + 4x^2 + 8x + 8 = 0$

24. $2x - \ln x - 7 = 0$

25. $x^2 - 1.25x + 0.25 = 0$

26. $x^2 + 5.15x + 2.37 = 0$

27. $x^4 - 11x + 8 = 0$

28. $\ln x - x + 3 = 0$

29. $e^x - x - 3 = 0$

30. $2x^2 - 12x + 11 = 0$

31. $x^4 - 5x^2 + x + 3 = 0$

32. $x^3 + 4x - 9 = 0$

33. $x^7 - 3x^6 + 3x^5 - 13x^4 + 43x^3 - 57x^2 + 33x - 7 = 0$

34. $\sin x - \frac{x}{2} = 0$ 35. $x^5 + x - 1000 = 0$ 36. $\sqrt{x} - \frac{1}{x} - 3 = 0$ 37. $\ln x + \sqrt{x} - 5 = 0$

38. $e^x \sin x - 2x - 5 = 0$

39. $e^{-x} - \cos x = 0$

40. $\cos^2 x - \frac{x}{5} = 0$

41. $(1 + \cos x)(e^x - 2) = 0$

42. $e^{-x} + \sin x - 1 = 0$

43. $xe^{-x} - 0.1 = 0$

44. $x^2 + \sin x + x = 0$

45. $\sin(2\cos x) - 1 - x^2 + e^{\sin^3 x} = 0$

46. $x^6 - 10x^3 + x^2 - x + 3 = 0$

47. $x^4 - x^3 + 11x - 7 = 0$

48. $x^3 - \cos x + 2 = 0$ 49. $\sqrt{x} - \cos x = 0$ 50. $\ln x - x^3 + 2\sin x = 0$

Problems 1-8 are from Noor [11], problems 9-34 are from Ndayawo and Sani [14], problems 35-38 are from Chun and Neta [15], problems 39-42 are from Kumar et al [16] and problems 42-50 are from Soleymani [17].

APPENDIX II. Comparison between number of iterations

	$\frac{f''(x)}{f'(x)} \approx m$	Initial point (x_0)	x_n (SOLUTION)	Noor	Nr	Chun	Basto	Ndayawo & Sani	New Method
1	1.781616671	1.3	1.404491649	4	4	3	2	3	3
2	0.8434823573	2	0.257530285	9	5	4	5	6	4
3	-0.06469185661	1.7	0.739085133	5	4	4	3	6	3
4	0.8	3.5	2	6	6	4	3	5	3
5	1.333333333	1.5	2.15443469	5	5	5	3	4	5
6	-4.874055511	-2	-0.915306601	5	4	6	DIV	DIV	5
7	-4.864176091	-2	-1.207647827	5	DIV	3	DIV	4	4
8	14.14285714	3.5	3	6	DIV	8	DIV	11	6
9	0.9500416252	-0.05	-0.31584581	4	4	2	3	3	3
10	- 1.942835793	0.86	1	5	4	2	3	4	6
11	-0.0736303089	0.11	0.111264158	2	2	1	2	2	1
12	0.7977207977	-0.35	-0.370252219	3	3	2	4	2	1
13	-0.8639030366	0.67	0.732050808	3	4	DIV	2	3	2
14	0.4077312657	2.5	2.883236873	DIV	3	3	3	4	3
15	-0.7406576142	0.33	0.347296355	3	3	2	2	3	2
16	0.01266303659	5.3	5.926476625	DIV	2	2	2	4	2
17	0.02359377230	1.5	1.523592933	5	3	1	2	3	2
18	-0.1666666667	2	1.557145599	7	3	2	3	5	2
19	0.6185567010	-0.1	-0.101031258	2	2	2	2	2	2
20	--0.3598004626	0.58	0.59368584	3	4	2	3	3	2
21	0.2553419212	-0.5	-0.251318625	7	3	2	3	4	2
22	-0..1192029220	2	2.120028239	5	3	3	2	3	2
23	0.6666666667	-1	-2	5	1	1	4	7	1
24	0.04761904762	3.5	4.219906484	DIV	3	3	2	4	3
25	- 2.352941176	0.2	0.25	DIV	3	2	3	4	2
26	0.4210526316	-0.2	-0.510871855	4	4	2	2	4	4
27	-0.6771332110	0.73	0.757149516	3	3	4	2	4	5
28	0.1666666667	3	4.505241496	DIV	5	3	3	7	3

	$\frac{f''(x)}{f'(x)} \approx m$	**Initial point** (x_0)	x_n **(SOLUTION)**	**Noor**	**Nr**	**Chun**	**Basto**	**Ndayawo & Sani**	**New Method**
29	-0.05239569650	-3	-2.947530903	6	2	1	2	3	2
30	-0.4807692308	0.92	1.129171307	5	4	2	3	5	2
31	- 1.272727273	-3	-2.198691243	7	6	3	4	6	4
32	0 .7035830619	2.25	1.464595701	4	5	3	3	4	3
33	- 4.639816912	0.21	0.593685833	DIV	7	5	3	8	5
34	0. .9925236770	2	1.895494267	3	3	2	2	3	2
35	0 .9992193599	4	3.977899394	3	3	2	2	2	2
36	- 1.500000000	1	9.633595562	DIV	5	3	DIV	DIV	4
37	-0.2559830602	3	8.309432693	DIV	5	3	5	DIV	3
38	-0.1883330467	-1	-2.52324523	9	3	2	3	6	3
39	0.3794947184	1.5	1.29269572	4	4	2	3	5	3
40	1.037575341	0.5	1.085982678	4	4	2	3	4	3
41	- 0.04148371676	0.9	0.693147181	7	3	2	3	4	2
42	0.8420313256	2.3	2.076831274	4	4	2	2	3	3
43	- 1.909090909	-0.1	0.111832559	5	4	3	3	5	3
44	0.6670275325	0.3	0.00E+00	DIV	5	3	DIV	DIV	4
45	-1.548924794	-0.82	0.8135737292	3	3	2	2	4	2
46	1.984922746	0.81	0.6586048471	6	DIV	3	3	4	3
47	0.3761100470	0.9	0.6450239555	5	3	2	4	5	3
48	- 2.244201985	-1.1	-1.172577964	4	4	2	3	4	3
49	0.07049049813	1.3	0.6417143709	DIV	4	2	3	6	2
50	2.254786488	1.4	1.297997743	4	DIV	2	4	DIV	3

DIV: Diverging
Newton Raphson's method: NR

APPENDIX III

If $x = x_0 = \gamma$ for initial guess

Then from equation (9), we have

$$N(x_0) = -m\frac{(x_0-\gamma)^2}{2} - \frac{g(x_0)}{f'(\gamma)} = -m\frac{(x_0-\gamma)^2}{2} - \frac{f(x_0)}{f'(\gamma)} \text{ at initial guess}$$

$$\Rightarrow N(x_0) = -m\frac{(x_0-x_0)^2}{2} - \frac{f(x_0)}{f'(\gamma)}$$

$$\therefore N(x_0) = x_1 = -\frac{f(x_0)}{f'(\gamma)}$$

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/67314