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Modified Adomian Decomposition Method for the Solution of Integro-Differential Equations

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Authors' contributions

This work was carried out in collaboration with all authors. OAT designed the study, ROI did the modification of the method and wrote the protocols, BKA wrote the first draft of the manuscript and managed the literature searches. All author read and approved the final manuscript.

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Abstract

This paper is concerned with modification of the Adomian Decomposition Method for solving linear and non-linear Volterra and Volterra-Fredholm Integro-Differential equations. The Modified form of ADM was carried out by replacing the Adomian polynomials constructed in the conventional Adomian Decomposition Method with the constructed canonical polynomials. The modified Adomian Decomposition Method was applied to solve some existing example. The results obtained using the newly modified ADM proved superior when compared with the conventional ADM.

Keywords: Integro-differential equation; approximate solution; differential equation; linear differential equation.

1 Introduction

An integro-differential equation is a mathematical expression which contains derivative of a required function and its integral transforms, such equations are typical of those processes where a quantity

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of interest that is a required function at each point is unambiguously determined by its value near the points described by differential equation but also depends on the function distributed all over the domain.

Integro-differential equations are important euations that have applications in the field of engineering, mechanics, physics, astronomy, potential theory and electrostatics among other areas. These equations are dificult to solve analytically, hence numerical approach are often applied. Many numerical methods have been developed in recent years for the solution of integro-differential equations, such methods include multistep method (Kajani and Gholampoor [1]), spectral collocation method (Doha, Abdelkwy and Amin [2]). Many authors have worked extensively on integro-differential equations (see Guslu and Sezer [3], Cao and Wang [4], Bhrawy et. al [5]).

Ine and Evans [6] applied the Adomian Decomposition Method (ADM) to solve singular ordinary differntial equations. Kumar and Singh [7], employed the new modified decomposition method. Behiry [8], applied the differential transformation method to high-order nonlinear Volterra-Fredholm integro-differential equations with separable kernels. Behzadi [9], solved a two-dimensional nonlinear Volterra-Fredholm integro differential equation by using the Modified Adomian Decomposition method, Variation iteration, Homotopy analysis method and Modified homotopy perturbation method.

1.1 General problem considered

In this paper, the basic ideas of the research work done by Adomian and Wazwaz were modified and applied to high-order non-linear Volterra-Fredholm integro-differential equation of the form:

$$\sum_{k=0}^{m} P_k y^{(k)}(x) = f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t) F_i(y(t))dt + \lambda_2 \int_a^b \sum_{i=0}^{s} B_j(x,t) G_j(y(t))dt \tag{1.1}$$

subject to the initial conditions

$$y^e(0) = \alpha_e, \qquad e = 0, 1, 2, \cdots, k-1 \tag{1.2}$$

where $P_k(x); (k = 0, 1, \cdots, m), A_i(x,t); (i = 0, 1, \cdots, r), B_j(x,t); (j = 0, 1, \cdots, s)$ and $f(x)$ are given functions, $y^{(k)}(x)$ indicates the $k-th$ derivative of $y(x)$, $F_i(y(x)), G_j(y(x))$ are non-linear functions, $\lambda_1, \lambda_2, \alpha_e; (e = 0, 1, \cdots, k-1)$ are real finite constants.

2 Adomain Decomposition Method

The Adomian decomposition method is a well-known systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations, to mention but few. The Adomian decomposition method is a powerful techniques, which provides efficient algorithms for analytic approximate solutions and numerical simulations for real-world applications in the applied sciences and engineering. It permits us to solve both nonlinear initial value problems (IVPs) and boundary value problems (BVPs) without unphysical restrictive assumptions such as required by linearization, perturbation, guessing the initial term or a set of basis functions, and so forth. Furthermore the Adomian decomposition method does not require the use of Green's functions, which would complicate such analytic calculations since Green's functions are not easily determined in most cases. The accuracy of the analytic approximate solution obtained can be verified by direct substitution.

This method originated from the book of Adomain. The idea of the method is to write the differential equation in the form

$$M[x, y(x)] = g(x) \tag{2.1}$$

Where g is a given function, y is unknown solution and M is a suitable operator. M is then decomposed accordingly as

$$M = L_1 + L_2 + N \tag{2.2}$$

Where L_1 and L_2 are linear operators chosen such that the inverse of L_1 can be easily found, while N denotes the nonlinear part. The method is based on the assumption that the exact solution y can be decomposed into a convergent series

$$y = \sum_{n=0}^{\infty} y_n \tag{2.3}$$

We also decompose the nonlinearity N in the form

$$Ny(x) = \sum_{n=0}^{\infty} A_x(x) \tag{2.4}$$

with the so called Adomain polynomial

$$A_n(x) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(x, \sum_{j=0}^{n} \lambda_j y_j\right)\right]_{\lambda=0} \tag{2.5}$$

A_n then depends on y_0 , $y_1 \cdots y_n$
Setting

$$y_0 = L_1^{-1} g \tag{2.6}$$

We proceed by the recurrence relation

$$y_n = L_1^{-1}(L_2 y_{n-1} - A_{n-1}), \quad n = 1, 2, \cdots \tag{2.7}$$

which defines the remaining terms of the series (2.7). Most authors dealing with this method assume that the series expansion of the equation (2.7) converges but do not look for conditions under which this assumption is satisfied.

2.1 Modified adomian decomposition method

In this section, we constructed canonical polynomials from equation (1.4) as follows: Let $m = 2$, we have

$$Ly : (p_2 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_0)y = r(x) \tag{2.8}$$

where L is a differential operator.

Let

$$LQ_j(x) = x^j \tag{2.9}$$

then

$$Lx^j = j(j-1)p_2 x^{j-2} + jp_1 x^{j-1} + p_0 x^j \tag{2.10}$$

implies

$$L(LQ_j(x)) = Lx^j \tag{2.11}$$

Hence,

$$L(LQ_j(x)) = p_2 j(j-1)x^{j-2} + p_1 j x^{j-1} + p_0 x^j, \tag{2.12}$$

$$L(LQ_j(x)) = p_2 j(j-1)LQ_{j-2}(x) + p_1 j LQ_{j-1}(x) + p_0 LQ_j(x) \tag{2.13}$$

Note that $LL^{-1} = I$ then equation (2.13) becomes

$$LQ_j(x) = p_2 j(j-1)Q_{j-2}(x) + p_1 j Q_{j-1}(x) + p_0 Q_j(x) \tag{2.14}$$

Thus,

$$x^j = p_2 j(j-1)Q_{j-2}(x) + p_1 jQ_{j-1}(x) + p_0 Q_j(x) \tag{2.15}$$

and

$$Q_j(x) = \frac{1}{p_0}(x^j - p_2 j(j-1)Q_{j-2}(x) - p_1 jQ_{j-1}(x)); j \geq 0 \tag{2.16}$$

For $j = 0$, we obtained from (2.16)

$$\begin{aligned}
& Q_0(x) = \frac{1}{p_0} \\
\text{For} j = 1: \quad & Q_1(x) = \frac{1}{p_0}(x - p_1 Q_0(x)) \\
\text{For} j = 2: \quad & Q_2(x) = \frac{1}{p_0}((x^2 - -2p_1 Q_1(x)) \\
\text{For} j = 3: \quad & Q_3(x) = \frac{1}{p_0}(x^3 - 6Q_1(x) - 3p_1 Q_2(x)) \\
\text{For} j = 4: \quad & Q_4(x) = \frac{1}{p_0}(x^4 - 12Q_2(x) - 4p_1 Q_3(x)) \\
\text{For} j = 5: \quad & Q_5(x) = \frac{1}{p_0}(x^5 - 20Q_3(x) - 5p_1 Q_4(x)) \\
\text{For} j = 6: \quad & Q_6(x) = \frac{1}{p_0}(x^6 - 30Q_4(x) - 6p_1 Q_5(x))
\end{aligned} \tag{2.17}$$

3 Demonstration of Adomian Decomposition Method on General Class of Problem Considered

$$\begin{aligned}
\sum_{k=0}^{m} P_k y^{(k)}(x) = f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(y(t))dt \\
+ \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(y(t))dt
\end{aligned} \tag{3.1}$$

Note

$$\begin{aligned}
\sum_{k=0}^{m} P_k y^{(k)}(x) = P_0(x)y(x) + P_1(x)y'(x) + P_2(x)y''(x) + \cdots + P_{m-1}(x)y^{m-1}(x) \\
+ P_m(x)y^m(x)
\end{aligned}$$

$$\begin{aligned}
\therefore \sum_{k=0}^{m-1} P_k y^{(k)}(x) + P_m(x)y^m(x) = f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(y(t))dt \\
+ \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(y(t))dt
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
P_m(x)y^m(x) = -\sum_{k=0}^{m-1} P_k y^{(k)}(x) + f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(y(t))dt \\
+ \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(y(t))dt
\end{aligned} \tag{3.3}$$

$$y^m(x) = \frac{-1}{P_m(x)} \left[\sum_{k=0}^{m-1} P_k y^{(k)}(x) + f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(y(t))dt + \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(y(t))dt \right] \tag{3.4}$$

In an operator form

$$y^m(x) = L^m y$$

$$y^{m-1}(x) = L^{m-1} y$$

$$\begin{aligned} L^m y = \frac{-1}{P_m(x)} \sum_{k=0}^{m-1} P_k L^{m-1} y + f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t) F_i(y(t)) dt \\ + \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t) G_j(y(t)) dt \end{aligned} \tag{3.5}$$

Operating with the inverse i.e. L^{-m} on both sides of equation (3.5), we have

$$\begin{aligned} L^{-m} L^m y = -L^{-m} \left[\frac{1}{P_m(x)} \sum_{k=0}^{m-1} P_k L^{m-1} y \right] + L^{-m} f(x) + L^{-m} \left[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t) F_i(y(t)) dt \right] \\ + L^{-m} \left[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t) G_j(y(t)) dt \right] \end{aligned} \tag{3.6}$$

$$\begin{aligned} y = -L^{-m} \left[\frac{1}{P_m(x)} \sum_{k=0}^{m-1} P_k L^{m-1} y \right] + L^{-m} f(x) + L^{-m} \left[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t) F_i(y(t)) dt \right] \\ + L^{-m} \left[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t) G_j(y(t)) dt \right] \end{aligned} \tag{3.7}$$

The Adomian decomposition method introduces the following expressions;

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{3.8}$$

The method defined the non-linear function as

$$F_i(y(x)) = \sum_{n=0}^{\infty} (C_i)_n \tag{3.9}$$

$$G_j(y(x)) = \sum_{n=0}^{\infty} (D_j)_n \tag{3.10}$$

$$L^{m-1} y = \sum_{n=0}^{\infty} E_n \tag{3.11}$$

Where $(C_i)_n, (D_j)_n$ and E_n are the approximate Adomian's polynomials.
Substituting (3.8), (3.9), (3.10), (3.11) into (3.7), we have

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(x) = -L^{-m} \left[\frac{1}{P_m(x)} \sum_{k=0}^{m-1} P_k \sum_{n=0}^{\infty} E_n \right] + L^{-m} f(x) + L^{-m} \left[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t) \sum_{n=0}^{\infty} (C_i)_n dt \right] \\ + L^{-m} \left[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t) \sum_{n=0}^{\infty} (D_j)_n dt \right] \end{aligned} \tag{3.12}$$

The remaining components of $y(x)$ is completely determined such that each term is computed by using the previous term as

$$y_{k+1}(x) = -L^{-m}\left[\frac{1}{P_m(x)}\sum_{k=0}^{m-1} P_k E_k\right] + L^{-m}f(x) + L^{-m}\left[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)(C_i)_k dt\right]$$
$$+ L^{-m}\left[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)(D_j)_k dt\right], \qquad k \geq 1$$

Note, the $n - term$ approximation is given as

$$\phi_n = \sum_{n=0}^{n-1} y_k \tag{3.13}$$

A new recursive scheme is formulated as follows

$$y_0 = \alpha_0$$
$$y_1 = \sum_{i=0}^{m-1} \alpha_1 \frac{1}{L!} x^L + L^{-m}f(x) - L^{-m}\left[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)(C_i)dt + \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)(D_j)dt\right]$$
$$y_{k+1}(x) = L^{-m}\left[\lambda_1 \int_0^x \sum_{i=0}^{r} A_i(x,t)(C_i)_k dt + \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)(D_j)_k dt\right], \qquad k \geq 1.$$

3.1 Demonstration of modified adomian decomposition method on general class of problem considered

$$\begin{aligned}\sum_{k=0}^{m} P_k y^{(k)}(x) &= f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(y(t))dt \\ &+ \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(y(t))dt\end{aligned} \tag{3.14}$$

Note

$$\sum_{k=0}^{m} P_k y^{(k)}(x) = P_0(x)y(x) + P_1(x)y'(x) + P_2(x)y''(x) + \cdots + P_{m-1}(x)y^{m-1}(x)$$
$$+ P_m(x)y^m(x)$$

$$\begin{aligned}\therefore \sum_{k=0}^{m-1} P_k y^{(k)}(x) + P_m(x)y^m(x) &= f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(y(t))dt \\ &+ \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(y(t))dt\end{aligned} \tag{3.15}$$

$$\begin{aligned}P_m(x)y^m(x) &= -\sum_{k=0}^{m-1} P_k y^{(k)}(x) + f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(y(t))dt \\ &+ \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(y(t))dt\end{aligned} \tag{3.16}$$

$$y^m(x) = \frac{-1}{P_m(x)}\left[\sum_{k=0}^{m-1} P_k y^{(k)}(x) + f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t)F_i(y(t))dt + \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t)G_j(y(t))dt\right] \tag{3.17}$$

In an operator form

$$y^m(x) = L^m y$$

$$y^k(x) = L^{m-1} y$$

$$L^m y = \frac{-1}{P_m(x)} \sum_{k=0}^{m-1} P_k L^{m-1} y + f(x) + \lambda_1 \int_a^x \sum_{i=0}^{r} A(x,t) F_i(y(t)) dt + \lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t) G_j(y(t)) dt \tag{3.18}$$

Operating with the inverse i.e. L^{-m} on both sides of equation (3.18), we have

$$L^{-m} L^m y = -L^{-m} \left[\frac{1}{P_m(x)} \sum_{k=0}^{m-1} P_k L^{m-1} y \right] + L^{-m} f(x) + L^{-m} \left[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t) F_i(y(t)) dt \right] + L^{-m} \left[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t) G_j(y(t)) dt \right] \tag{3.19}$$

$$y = -L^{-m} \left[\frac{1}{P_m(x)} \sum_{k=0}^{m-1} P_k L^{m-1}(y) \right] + L^{-m} f(x) + L^{-m} \left[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t) F_i(y(t)) dt \right] + L^{-m} \left[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t) G_j(y(t)) dt \right] \tag{3.20}$$

The Adomian decomposition method introduces the following expressions;

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{3.21}$$

The method defined the non-linear function as

$$F_i(y(x)) = \sum_{n=0}^{\infty} (C_i)_n \tag{3.22}$$

$$G_j(y(x)) = \sum_{n=0}^{\infty} (D_j)_n \tag{3.23}$$

$$L^{m-1} y = \sum_{n=0}^{\infty} E_n \tag{3.24}$$

Where $(C_i)_n, (D_j)_n$ and E_n are the Canonical polynomials generated in equation(2.17).
Substituting (3.21), (3.22), (3.23), (3.24) into (3.20), we have

$$\sum_{n=0}^{\infty} y_n(x) = -L^{-m} \left[\frac{1}{P_m(x)} \sum_{k=0}^{m-1} P_k \sum_{n=0}^{\infty} E_n \right] + L^{-m} f(x) + L^{-m} \left[\lambda_1 \int_a^x \sum_{i=0}^{r} A_i(x,t) \sum_{n=0}^{\infty} (C_i)_n dt \right] + L^{-m} \left[\lambda_2 \int_a^b \sum_{j=0}^{s} B_j(x,t) \sum_{n=0}^{\infty} (D_j)_n dt \right] \tag{3.25}$$

The remaining components of $y(x)$ is completely determined such that each term is computed by using the previous term as

$$y_{k+1}(x) = -L^{-m}\left[\frac{1}{P_m(x)}\sum_{k=0}^{m-1}P_kE_k\right] + L^{-m}\left[\lambda_1\int_a^x\sum_{i=0}^{r}A_i(x,t)(C_i)_k dt\right]$$
$$+ L^{-m}\left[\lambda_2\int_a^b\sum_{j=0}^{s}B_j(x,t)(D_j)_k dt\right], \qquad k \geq 1$$

Note, the $n - term$ approximation is given as

$$\phi_n - \sum_{n=0}^{n-1} y_k \tag{3.26}$$

A new recursive scheme is formulated as follows

$$y_0 = \alpha_0$$
$$y_1 = \sum_{i=0}^{m-1}\alpha_1\frac{1}{L!}x^L + L^{-m}f(x) - L^{-m}\left[\lambda_1\int_a^x\sum_{i=0}^{r}A_i(x,t)(C_i)dt + \lambda_2\int_a^b\sum_{j=0}^{s}B_j(x,t)(D_j)dt\right]$$
$$y_{k+1}(x) = L^{-m}\left[\lambda_1\int_a^x\sum_{i=0}^{r}A_i(x,t)(C_i)_k dt + \lambda_2\int_a^b\sum_{j=0}^{s}B_j(x,t)(D_j)_k dt\right], \qquad k \geq 1.$$

4 Numerical Experiments

4.1 Demonstration of modified adomian decomposition method on an example

Example 1

$$y'(x) + 2xy(x) = f(x) + \int_0^x (x+t)y^3(t)dt + \int_0^1 (x-t)y(t)dt \tag{4.1}$$

where

$$f(x) = \left(\frac{-2}{3}x + \frac{1}{9}\right)e^{3x} + (2x+1)\,e^x + \left(\frac{4}{3} - e\right)x + \frac{8}{9}$$

with condition $y(0) = 1$. Here comparing equations (4.5) with equations (3.1) and (3.2),we obtain $a = 0$, $b = 1$, $p_1(x) = 1$, $p_2(x) = 2x$ and the exact solution is $y(x) = e^x$

Solution

In an operator form

$$y'(x) = -2xy(x) + f(x) + \int_0^x (x+t)y^3(t)dt + \int_0^1 (x-t)y(t)dt$$

Note

$$L^1 = \frac{d}{dx} \quad L^{-1} = \int_0^x$$

$$L^1y(x) = -2xy(x) + f(x) + \int_0^x (x+t)y^3(t)dt + \int_0^1 (x-t)y(t)dt$$

$$L^{-1}L^1y(x) = -2L^{-1}xy(x) + L^{-1}f(x) + L^{-1}\int_0^x (x+t)y^3(t)dt + L^{-1}\int_0^1 (x-t)y(t)dt$$

$$\int_0^x y'(x) = -2L^{-1}xy(x) + L^{-1}f(x) + L^{-1}\int_0^x (x+t)y^3(t)dt + L^{-1}\int_0^1 (x-t)y(t)dt$$

$$y(x)|_0^x = -2L^{-1}xy(x) + L^{-1}f(x) + L^{-1}\int_0^x (x+t)y^3(t)dt + L^{-1}\int_0^1 (x-t)y(t)dt$$

$$y(x) - y(0) = -2L^{-1}xy(x) + L^{-1}f(x) + L^{-1}\int_0^x (x+t)y^3(t)dt + L^{-1}\int_0^1 (x-t)y(t)dt$$

but $y(0) = 1$

$$y(x) = 1 - 2L^{-1}xy(x) + L^{-1}f(x) + L^{-1}\int_0^x (x+t)y^3(t)dt + L^{-1}\int_0^1 (x-t)y(t)dt$$

Let $y(x) = \sum_{n=0}^{\infty} y_n(x)$

$F(y(t)) = y^3(t)$

$\therefore \quad F(y(t)) = \sum_{n=0}^{\infty} C_n(t)$

which now becomes

$$\sum_{n=0}^{\infty} y_n(x) = 1 - 2L^{-1}x\sum_{n=0}^{\infty} y_n(x) + L^{-1}f(x) + L^{-1}\int_0^x (x+t)\sum_{n=0}^{\infty} C_n(t)dt + L^{-1}\int_0^1 (x-t)\sum_{n=0}^{\infty} y_n(x)dt$$

where $C_n(t)$ are the canonical polynomials generated in equation (3.19) above
but

$$y_0(t) = y(0) = 1$$

$$y_{k+1} = -2L^{-1}xy_k(x) + L^{-1}f(x) + L^{-1}\left[\int_0^x (x+t)C_k(t)dt + \int_0^1 (x-t)y_k(x)dt\right]$$

when $k = 0$

$$y_1 = -2L^{-1}xy_0(x) + L^{-1}f(x) + L^{-1}\left[\int_0^x (x+t)C_0(t)dt + \int_0^1 (x-t)y_0(x)dt\right]$$

when $k \geq 1$

$$y_{k+1} = L^{-1}\left[\int_0^x (x+t)C_k(t)dt + \int_0^1 (x-t)y_k(x)dt\right]$$

Recall that $L^1(\cdot) = \int_0^x (\cdot)dx$

$$\begin{aligned} y_0 &= 1 \\ y_1 &= -2L^{-1}xy_0(x) + L^{-1}\left[\left(\frac{-2}{3}x + \frac{1}{9}\right)e^{3x} + (2x+1)e^x + \left(\frac{4}{3} - e\right)x + \frac{8}{9}\right] \\ &\quad + L^{-1}\left[\int_0^x (x+t)C_0(t)dt + \int_0^1 (x-t)y_0(x)dt\right] \end{aligned}$$

$$\begin{aligned} C_0(t) &= 1 \\ C_1(t) &= t - 1 \\ C_2(t) &= t^2 - 2t + 2 \end{aligned}$$

$$\begin{aligned} y_1 &= -2\int_0^x xdx + \int_0^x \left[\left(\frac{-2}{3}x + \frac{1}{9}\right)e^{3x} + (2x+1)e^x + \left(\frac{4}{3} - e\right)x + \frac{8}{9}\right]dx \\ &\quad + \int_0^x \int_0^x (x+t)dx + \int_0^x (x-t)dx \end{aligned}$$

$$y_1 = \frac{1}{6}x^2 + \frac{8}{9} - \frac{2}{9}e^{3x} + \frac{1}{9}e^{3x} + 2e^x x - e^x - \frac{1}{2}x^2 e + \frac{7}{18}x + \frac{1}{2}x^3$$

when $k = 1$

$$y_2 = L^{-1}\left[\int_0^x (x+t)(t-1)dt + \int_0^1 (x-t)y_1(t)dt\right]$$

$$= \frac{5\,x^4}{24} - 1/2\,x^3 + 1/2\left(\frac{2723}{648} - 7/6\,\mathrm{e} - \frac{\mathrm{e}^3}{81}\right)x^2 + \frac{41363\,x}{9720} + \frac{4\,x\mathrm{e}^3}{243} - \frac{15\,x\mathrm{e}}{8}$$

$$y_3 = L^{-1}\left[\int_0^x (x+t)(t^2-2t+2)dt + \int_0^1 (x-t)y_2(t)dt\right]$$

$$= -\frac{x^5}{240} - \frac{5\,x^4}{12} + x^3 + \frac{5\,x^6}{144} + 1/4\left(\frac{3047}{1296} - \frac{7\,\mathrm{e}}{12} - \frac{\mathrm{e}^3}{162}\right)x^4 + 1/3\left(\frac{124607}{38880} - \frac{19\,\mathrm{e}}{12} + \frac{19\,\mathrm{e}^3}{972}\right)x^3 + 1/2\left(-\frac{41363}{19440} - \frac{2\,\mathrm{e}^3}{243} + \frac{15\,\mathrm{e}}{16}\right)x^2$$

The approximate solution is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) \tag{4.2}$$

$$y(x) = \frac{17}{9} + 1/6\,x^2 - 2/9\,\mathrm{e}^{3\,x}x + 1/9\,\mathrm{e}^{3\,x} + 2\,\mathrm{e}^x x - \mathrm{e}^x - 1/2\,x^2\mathrm{e} + \frac{45143\,x}{9720} + x^3 - \frac{5\,x^4}{24} + 1/2\left(\frac{2723}{648} - 7/6\,\mathrm{e} - \frac{\mathrm{e}^3}{81}\right)x^2 + \frac{4\,x\mathrm{e}^3}{243} - \frac{15\,x\mathrm{e}}{8} - \frac{x^5}{240} + \frac{5\,x^6}{144} + 1/4\left(\frac{3047}{1296} - \frac{7\,\mathrm{e}}{12} - \frac{\mathrm{e}^3}{162}\right)x^4 + 1/3\left(\frac{124607}{38880} - \frac{19\,\mathrm{e}}{12} + \frac{19\,\mathrm{e}^3}{972}\right)x^3 + 1/2\left(-\frac{41363}{19440} - \frac{2\,\mathrm{e}^3}{243} + \frac{15\,\mathrm{e}}{16}\right)x^2$$

Demonstration of Modified Adomian Decomposition Method on Example 2

Let us consider the Votterra integro-differential equation

$$y''(x) + xy(x) = f(x) + \int_0^x x^2 e^x y(t)dt \tag{4.3}$$

where

$$f(x) = -\cos x + x\cos(x) - x^2\left(\frac{-1}{2}x + \frac{e^x(\cos x) + \sin(x)}{2}\right)$$

with condition $y(0) = 1$, $y'(0) = 0$ and the exact solution is $y(x) = \cos x$. Here comparing equations (4.7) with equations (3.1) and (3.2),we obtain$a = 0$, $b = 1$, $p_0(x) = x$, $p_1(x) = 0$, $p_2(x) = 1$

Solution

In an operator form, equation (4.7) becomes

$$L^2 y(x) = -xy(x) + f(x) + \int_0^x x^2 e^x y(t)dt$$

where $L^2 = y''(x)$

Applying L^{-2}, it becomes

$$L^{-2}L^2y(x) = -L^{-2}[xy(x)] + L^{-2}f(x) + L^{-2}\left[\int_0^x x^2e^xy(t)dt\right]$$

$$y(x) = y(0) - y'(0)x + L^{-2}f(x) + L^{-2}\left[\int_0^x x^2e^xy(t)dt\right] - L^{-2}[xy(x)]$$

using the conditions $y(0) = 1$, $y'(0) = 0$, we have

$$y(x) = 1 + L^{-2}f(x) + L^{-2}\left[\int_0^x x^2e^xy(t)dt\right] - L^{-2}[xy(x)]$$

Substituting the decomposition series $y(x) = \sum_{n=0}^{\infty} y_n(x)$ for $y(x)$ yields

$$\sum_{n=0}^{\infty} y_n(x) = 1 + L^{-2}f(x) + L^{-2}\left[\int_0^x x^2e^x\left(\sum_{n=0}^{\infty} y_n(x)\right)dt\right] - L^{-2}[xy(x)]$$

$$y_0(x) = 1$$

$$y_1 = L^{-2}\left[f(x) + \int_0^x x^2e^xc_0(t)dt\right] - L^{-2}[xy_0(x)]$$

$$-7/4 - 3/2\,x + \cos(x) + 2\,\sin(x) - x\cos(x) + 1/24\,x^4 + 3/4\,e^x\cos(x) + 3/4\,e^x\sin(x) + 1/4\,e^x\cos(x)\,x^2 - e^x\cos(x)\,x - 1/4\,e^x\sin(x)\,x^2 + x^5e^x - 1/6\,x^3$$

$$-\frac{2889\,x}{4} + \cos(x)\,x^2 - 1/8\,x^4 - \frac{7\,x^3}{24} + 1800\,x^2e^x + x^6e^x - 6\,x\sin(x) - 4320\,e^xx - 12\,x^5e^x + 90\,x^4e^x - 480\,x^3e^x - x\cos(x) + \frac{15\,e^x\cos(x)}{4} + 1/8\,e^x\cos(x)\,x^3 + 1/8\,e^x\sin(x)\,x^3 - 5/4\,e^x\sin(x)\,x^2 + 5/2\,e^x\sin(x)\,x - 5/2\,e^x\cos(x)\,x - \frac{20135}{4} + \frac{x^7}{1008} - \frac{x^6}{180} + 5040\,e^x - 10\,\cos(x) + 2\,\sin(x)\;57268\,x + 52\,x\cos(x) + 287\,x^5e^x + \frac{45\,e^x\cos(x)}{4} - \frac{45\,e^x\sin(x)}{4} - 10\,\cos(x) - 80\,\sin(x) - \frac{963\,x^4}{16} - 277200\,e^x - \frac{20135\,x^3}{24} - \frac{17\,e^x\sin(x)\,x^2}{4} - \frac{17\,e^x\cos(x)\,x^2}{4} - \frac{7\,x^6}{720} - \frac{x^7}{336} + \frac{31\,e^x\sin(x)\,x}{2} + \frac{9\,e^x\cos(x)\,x^3}{8} - \frac{x^9}{12960} + \frac{x^{10}}{90720} + \frac{1108795}{4} - 19\,x^6e^x + 219960\,e^xx - 6\,x\sin(x) - 2790\,x^4e^x + 18420\,x^3e^x - 81360\,x^2e^x + \cos(x)\,x^2 + 12\,\sin(x)\,x^2 - 1/16\,e^x\cos(x)\,x^4 + 1/16\,e^x\sin(x)\,x^4 - \cos(x)\,x^3 + 5/6\,x^7e^x$$

The approximate solution is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) \tag{4.4}$$

$y(x) = (231955/4) \times x + 52 \times x \times cos(x) + 299 \times x^5 \times exp(x) + (33/4) \times exp(x) \times cos(x) - (21/2) \times exp(x) \times sin(x) + cos(x) - 80 \times sin(x) - (2881/48) \times x^4 - 282240 \times exp(x) - (5033/6) \times x^3 + (3/2) \times exp(x) \times cos(x) \times x - (13/4) \times exp(x) \times sin(x) \times x^2 - 4 \times exp(x) \times cos(x) \times x^2 - (1/240) \times x^6 - (1/252) \times x^7 - (1/8) \times exp(x) \times sin(x) \times x^3 + 13 \times exp(x) \times sin(x) \times x + exp(x) \times cos(x) \times x^3 - (1/12960) \times x^9 + (1/90720) \times x^10 - (39/2) \times x^6 \times exp(x) + 224280 \times exp(x) \times x - 2880 \times x^4 \times exp(x) + 18900 \times x^3 \times exp(x) - 83160 \times x^2 \times exp(x) + 12 \times sin(x) \times x^2 - (1/16) \times exp(x) \times cos(x) \times x^4 + (1/16) \times exp(x) \times sin(x) \times x^4 - cos(x) \times x^3 + (5/6) \times x^7 \times exp(x) + 1128927/4$

5 Table of Results

Table 1. Approximate solutions of *ADM* and *MADM* methods

x	Exact Solution	ADM SOLUTION	MADM SOLUTION
0	1	1	1
0.1	1.105170918	1.106584571	1.105099699
0.2	1.221402758	1.232265285	1.226644901
0.3	1.349858808	1.38138354	1.370958679
0.4	1.491824698	1.555642294	1.544556869
0.5	1.648721271	1.752621335	1.752602717
0.6	1.822118800	1.996767498	1.963593484
0.7	2.013752707	2.271661348	2.170330117
0.8	2.225540928	2.559078232	2.340455296
0.9	2.459603111	2.420727359	2.818867744
1.0	2.718281828	2.3273762	2.974691362

Table 2. Absolute error of example 1

x	Absolute Error for MADM	Absolute Error for ADM
0	0	0
0.1	1.4137×10^{-3}	7.1219×10^{-5}
0.2	1.0863×10^{-2}	5.2421×10^{-3}
0.3	3.1525×10^{-2}	2.1100×10^{-2}
0.4	6.3818×10^{-2}	5.2732×10^{-2}
0.5	1.0390×10^{-1}	1.0388×10^{-1}
0.6	1.7465×10^{-1}	1.1415×10^{-1}
0.7	2.5791×10^{-1}	1.5658×10^{-1}
0.8	3.3354×10^{-1}	1.1150×10^{-1}
0.9	3.8876×10^{-1}	3.5926×10^{-1}
1.0	3.9091×10^{-1}	2.5641×10^{-1}

Table 3. Solution OF example 2

x	Exact solution	ADM SOLUTION	MADM SOLUTION
0	1	1	1
0.1	0.9999984769	0.9949635001	0.995035117
0.2	0.9999939077	0.980428129	0.9811427381
0.3	0.9999862922	0.9611371238	0.958049452
0.4	0.9999756307	0.9486963588	0.933130990
0.5	0.9999619231	0.9702455385	0.916434756
0.6	0.9999451694	0.927343582	1.078590085
0.7	0.9999253697	1.368195179	0.997532716
0.8	0.9999025240	1.997170465	1.175560711
0.9	0.9998766325	3.222169017	1.532907813
1.0	0.9998476952	2.170869744	5.445197827

Table 4. Absolute error for example 2

x	ADM ABSOLUTE ERROR	MADM ABSOLUTE ERROR
0	0	0
0.1	5.03498×10^{-3}	4.9634×10^{-3}
0.2	1.9566×10^{-2}	1.8851×10^{-2}
0.3	3.8849×10^{-2}	4.1937×10^{-2}
0.4	5.1279×10^{-2}	6.6845×10^{-2}
0.5	2.9716×10^{-2}	8.3527×10^{-2}
0.6	7.2602×10^{-2}	7.8645×10^{-3}
0.7	2.3927×10^{-3}	3.6827×10^{-1}
0.8	9.9727×10^{-1}	1.7566×10^{-1}
0.9	2.2223×10^{1}	5.3303×10^{-1}
1.0	1.1710×10^{1}	4.4454×10^{1}

6 Conclusion

In this paper, Adomian decomposition method and Modified Adomian decomposition method was used to solve linear and non-linear Volterra-Fredholm integro-differential equations. From the tables of results, we observed that Modified Adomian decomposition method is more efficient, reliable and less computational in terms of cost.

Competing Interests

Authors have declared that no competing interests exist.

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