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Galileo's Parabola Observed from Pappus' Directrix, Apollonius' Pedal Curve (Line), Galileo's Empty Focus, Newton's Evolute, Leibniz's Subtangent and Subnormal, Ptolemy's Circle (Hodograph), and Dürer-Simon Parabola (16.03.2019)

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Abstract

Galileo's Parabola describing the projectile motion passed through hands of all scholars of the classical mechanics. Therefore, it seems to be impossible to bring to this topic anything new. In our approach we will observe the Galileo's Parabola from Pappus' Directrix, Apollonius' Pedal Curve (Line), Galileo's Empty Focus, Newton's Evolute, Leibniz's Subtangent and Subnormal, Ptolemy's Circle (Hodograph), and Dürer-Simon Parabola. For the description of events on this Galileo's Parabola (this conic section parabola was discovered by Menaechmus) we will employ the interplay of the directrix of parabola discovered by Pappus of Alexandria, the pedal curve with the pedal point in the focus discovered by Apollonius of Perga (The Great Geometer), and the Galileo's empty focus that plays an important function, too. We will study properties of this MAG Parabola with the aim to extract some hidden parameters behind that visible parabolic orbit in the Aristotelian World. For the visible Galileo's Parabola in the Aristotelian World, there might be hidden curves in the Plato's Realm behind the mechanism of that Parabola. The analysis of these curves could reveal to us hidden properties describing properties of that projectile motion. The parabolic path of the projectile motion can be described by six expressions of projectile speeds. In the Dürer-Simon's Parabola we have determined tangential and normal accelerations with resulting acceleration g = 9.81 msec^{-2} directing towards to Galileo's empty focus for the projectile moving to the vertex of that Parabola. When the projectile moves away from the vertex the resulting acceleration g = 9.81 msec^{-2} directs to the center of the Earth (the second focus of Galileo's Parabola in the "infinity"). We have extracted some additional properties of Galileo's Parabola. E.g., the Newtonian school correctly used the expression for "kinetic energy $E = ½ mv^2$ for parabolic orbits and paths, while the Leibnizian school correctly used the expression for "vis viva" $E = mv^2$ for hyperbolic orbits and paths. If we will insert the "vis viva" expression into the Soldner's formula (1801) (e.g., Fengyi Huang in 2017), then we will get the right experimental value for the deflection of light on hyperbolic orbits. In the Plato's Realm some other curves might be hidden and have been waiting for our future research. Have we found the Arriadne's Thread leading out of the Labyrinth or are we still lost in the Labyrinth?

Keywords: Galileo's Parabola, Aristotelian World, Plato's Realm, Hidden Mathematical Objects, Pappus' Directrix, Apollonius' Pedal Curves, MAG Parabola, Galileo's Empty Focus, Newton's Evolute, Leibniz's Subtangent and Subnormal, Ptolemy's Circle, Dürer-Simon Parabola, "Vis Viva" Controversy

1. Introduction

The famous quote of Heraclitus "Nature loves to hide" was described in details by Pierre Hadot in 2008. Hadot in his valuable book gives us many examples how Nature protects Her Secrets. In several situations the enormous research of many generations is strongly needed before the right "recipe" unlocking the true reality can be found. Johann Wolfgang Goethe remarked to our research: "Nature does not suffer Her veil to be taken from Her, and what She does not choose to reveal to the spirit, thou wilt wrest from Her by levers and screws."

Conic sections-Circle, Ellipse, Hyperbola, and Parabola-are among the oldest curves, and belong to the Treasure of Geometry. The conic sections seem to have been discovered by Menaechmus and were thoroughly studied by Apollonius of Perga (The Great Geometer) and his scholars. The conics are endowed with numerous beautiful properties, some those properties are shared by the entire family, while some properties are unique and original for each of them. In our research we have to be very careful and patient when we want to apply those properties for the projectile motion. Menaechmus said to Alexander the Great: "O King, for travelling over the country, there are royal roads and roads for common citizens, but in geometry there is one road for all."

Parabola is a very original conic section with its own Beauty and Secrets. Though, it has only one focus, it might reveal similar properties as Her Sisters Ellipse and Hyperbola. Pappus of Alexandria discovered the directrix and focus of the parabola, and Apollonius of Perga systematically revealed numerous properties of the parabola. This Ancient Treasure passed into the hands of Dürer, Copernicus, Galileo, Kepler, Huygens, Newton, Leibniz and many others. We have found that the empty focus of Galileo's parabola plays a significant role in the events on the Galileo's parabolic path, too. It is said that the second focus of parabola lays in the infinity. In our approach we will put the second focus of Galileo's parabola into the center of the Earth. In our approach both foci contribute to the resulting parabolic path.

W. R. Hamilton in 1847 discovered how to find the tangential velocity for the elliptic orbit using the auxiliary circle of that ellipse. This technique works very well for hyperbola with two foci, too. We have found that the Galileo's empty focus might be used for the determination of the tangential velocity of projectile motion, too.

A possible chance for further classical development of the Galileo's parabola is to penetrate more deeply into the secrets of the Galileo's parabola and to reappear with some new hidden properties overlooked by earlier generations of researchers. Our guiding principle is the existence of the Plato's Realm with invisible mathematical objects that might bring to us some additional information about the visible Galileo's parabola in the Aristotelian World. In this contribution we have been working with these mathematical objects from the Plato's Realm:

1. Parabola properties discovered by Apollonius of Perga-the Great Geometer-and many his scholars.

2. Directrix and focus of parabola discovered by Pappus of Alexandria

3. Locus of the radii of curvature (evolute) of parabola-Isaac Newton in 1687.

4. The interplay of parabola properties M (= Menaechmus the discoverer of the parabola): the directrix of parabola (Pappus-the discoverer of directrix), the vertex of the parabola A (= Apollonius of Perga-the Great Geometer), and the Galileo's empty focus
G (= Galileo-the Great Physicist) together form the MAG Parabola with several interesting hidden properties of those parabolic paths.

5. Pedal curve with the pedal point in the Galileo's empty focus-the "auxiliary circle"-the line perpendicular to the axis of parabola in the vertex.

6. Tangent and normal to a point on the parabolic orbit described by six parameters introduced by Gottfried Wilhelm Leibniz: abscissa, ordinate, length of tangent, subtangent, length of normal, and subnormal.

7. Moment of tangent momentum, moment of normal momentum.

8. Hodograph: Ptolemy's Circle with the diameter a (distance between the vertex of parabola and its focus) where we see the normal velocity of that projectile. In this case the Ptolemy's Circle plays a role of the hodograph rotating on the parabolic orbit without sliding.

9. Dürer-Simon Parabola-might reveal the tangential, normal and total accelerations of that moving projectile before the vertex and after the vertex of the Galileo's parabola.

The experimental analysis of properties of the MAG Parabola should reveal if we have found the Arriadne's Thread leading out of the Labyrinth or if we are still lost in the Labyrinth.

(We are aware of the famous quote of Richard Feynman from the year 1962: "There's certain irrationality to any work in gravitation, so it is hard to explain why you do any of it.")

2. Some Properties of the MAG Parabola

Figure 1, Figure 2, and Figure 3 show some parabolic properties of the MAG parabola that might be used for the description of motion of projectile around the Galileo's empty focus on the parabolic path. Some of those properties are very well-known from many good books on conic sections, some properties are newly derived. The great inspiration was found in the works of Isaac Todhunter (1881) who studied very deeply the properties of conic sections.

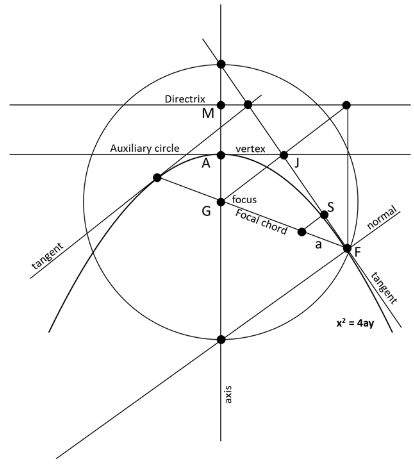

Figure 1. Some properties of the MAG Parabola: directrix M (=Menaechmus), vertex A (=Apollonius) with the “auxiliary circle”-the vertex line, Galileo’s empty focus G (=Galileo), tangent and normal to the point F on that parabola, focal chord

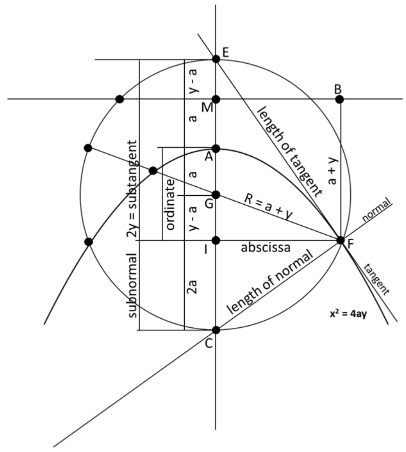

Figure 2. Some properties of the MAG Parabola with MAGIC Circle, and six Leibniz’s parameters: abscissa, ordinate, length of tangent, subtangent, length of normal, subnormal for the point F on the parabolic path

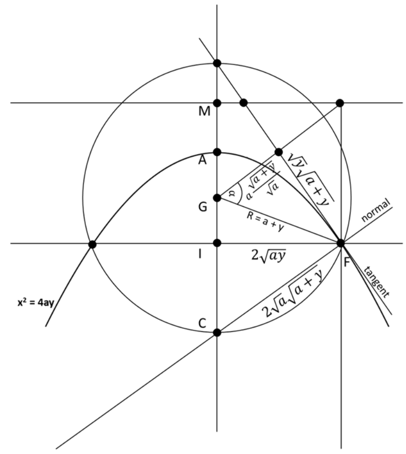

Figure 3. Some properties of the MAG Parabola for the quantitative calculation

Table 1 summarizes some relations for the MAG Parabola.

Table 1. Some properties of the MAG Parabola

Some properties of the MAG parabola $x^2 = 4\ ay$
$a = 1 = MA = AG$ … parameter of the MAG parabola
Latus rectum = 4a
MAGIC Circle for the construction of the tangent and the normal to a point F
M … Pappus' directrix
A … Apollonius' vertex
G … Galileo's empty focus
F … object on the parabolic path
J, S … points on the tangent
ρ … radius of curvature of parabola
Leibniz's six parameters of a point on the parabolic orbit
$Abscissa \quad x = 2\sqrt{ay}$
$Ordinate \quad y$
$Length\, of\, the \tan gent = 2\sqrt{y}\sqrt{a+y}$
$Sub \tan gent = 2y$
$Length\, of\, the\, normal = 2\sqrt{a}\sqrt{a+y}$
$Subnormal = 2a$
$GF = GC = a + y$
SURF = Ptolemy's Circle
$GJ = a\dfrac{\sqrt{a+y}}{\sqrt{a}}$
$SU = a\dfrac{\sqrt{a}}{\sqrt{a+y}}$
$SF = a\dfrac{\sqrt{y}}{\sqrt{a+y}}$
$\rho = \dfrac{2\left(a+y\right)^{3/2}}{\sqrt{a}}$

3. Classical Properties of the MAG Parabola

Projectile motion is a form of motion experienced by a projectile that is thrown near the Earth's surface and moves along the Galileo's Parabola under the action of gravity only (the effect of air resistance is neglected). Galileo decomposed the velocity of that projectile into the horizontal component that remains unchanged throughout the motion and into the vertical component that changes linearly because the acceleration due to gravity is constant. For ideal projectile motion when a projectile is launched from the original position over a horizontal surface with initial speed v_{T0} and launch angle α there are very well-known properties summarized in Table 2.

Figure 4 summarizes those very well-known parameters of the projectile motion: total time of the flight, maximum height of projectile (A-vertex of that parabola), range F'F-maximum distance of projectile, M-directrix as the measure of the total energy $E_{PAR} = K_{KIN} + E_{POT}$ of that projectile on the parabolic path, parameter *a* of that parabola describing kinetic energy of projectile at the vertex, G-Galileo's empty focus with some significant properties. We have added into the Figure 4 the point D describing the "vis viva" for the hyperbolic paths in order to compare the energy of parabolic and hyperbolic paths-see the last chapter of this contribution.

Table 2. Properties of the projectile motion on the Galileo's Parabola

Time of the flight T
$T = \dfrac{2\ v_{T0} \sin\alpha}{g}$
Range of the flight R
$R = \dfrac{v_{T0}^2 \sin(2\alpha)}{g}$
Parameter of the Galileo's Parabola a
$a = \dfrac{v_{T0}^2 \cos^2\alpha}{2g}$
Height of the "vis viva" for hyperbolic paths-D
$D = \dfrac{v_{T0}^2}{g}$
Height of the Subtangent E
$E = \dfrac{v_{T0}^2 \sin^2\alpha}{g}$
Height of the Directrix M-the energy for parabolic paths
$M = \dfrac{v_{T0}^2}{2g}$
Height of the Vertex A
$A = \dfrac{v_{T0}^2 \sin^2\alpha}{2g}$
Height of the Focus G
$G = \dfrac{v_{T0}^2 \cos(2\alpha)}{2g}$

4. Proposed Reflecting and Refracting Properties of Solar and Object Gravitons on the Parabolic Path

In this section we will assume that Earth's gravitons enter into the internal volume of the flying projectile on the parabolic path and collide with projectile gravitons in four possible scenarios as it was in details described by Jiří Stávek (2018). Gravitons are reflected and refracted similarly as in the case of the Kepler's ellipse, Newton's Hyperbola, and Newton's Parabola. Therefore, we will present here only a qualitative schema of reflective and refractive properties of parabola in Figure 5.

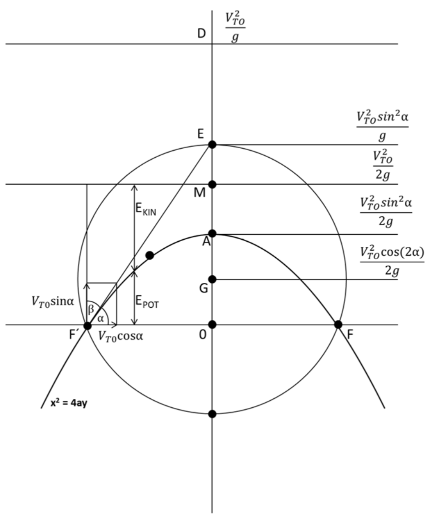

Figure 4. Properties of the MAG Parabola used in standard textbooks of the classical mechanics

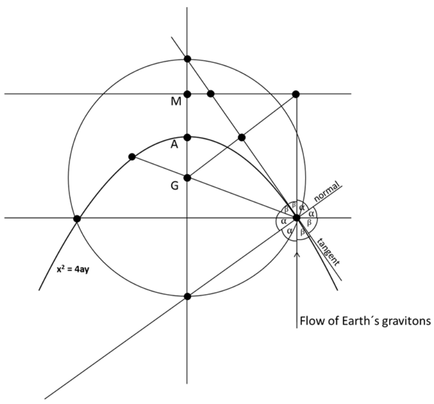

Figure 5. Reflective and refractive properties of the MAG parabola

5. Galileo's Parabola Observed from the Newton's Evolute

Newton discovered several important properties hidden in the Kepler's ellipse, the Newton's Hyperbola and the Newton's Parabola in his Principia in 1687. For the centripetal force F, he derived formula:

$$F = m \frac{v_T^2}{\rho} \tag{1}$$

where m is the mass of the planet (or any object), v_T is the tangent velocity of the planet and ρ is the radius of curvature of that ellipse, hyperbola or parabola. The locus of radii of curvature is termed as the evolute. This equation opened completely new possibilities in the understanding of the Kepler's ellipse, Newton's Hyperbola, and Newton's Parabola. We will use this procedure for the description of events in Galileo's parabola.

In the standard procedure both quantities v_T and ρ are found by the infinitesimal calculus discovered independently by Newton and Leibniz.

We will present here the trigonometric approach to these two quantities (v_T in the next chapter). The radius of curvature of the parabola ρ can be derived in the trigonometric way shown in Figure 6. Figure 6 describes an interplay between the normal to the tangent and the line connecting the center of the Earth and moving projectile.

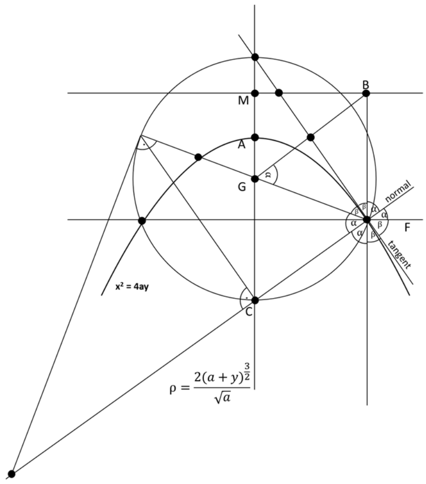

Figure 6. Trigonometric approach to reveal the expression for the radius of curvature ρ of the Galileo's parabola

We have used the deep knowledge of parabola properties of Issac Todhunter (1881) and extracted from Figure 6 the expression for the radius of curvature, which is already known in the existing literature:

$$\rho = \frac{2\left(a+y\right)^{3/2}}{\sqrt{a}} \tag{2}$$

(The quantities expressed in the trigonometric language are simpler and Nature can talk with us in this trigonometric language that could be depicted in simple Figures without words).

6. Galileo's Parabola Observed from the Pedal Curve with Pedal Point in the Galileo's Empty Focus and from the Leibniz's Subnormal

The pedal curve of the Galileo's parabola is the locus of the feet of the perpendiculars from the Galileo's empty focus to the tangent of that parabola. In this case the pedal curve is the famous "auxiliary circle" of the parabola-the line perpendicular to the axis of parabola in the vertex of that parabola. We will use the Hamilton's old very well-known recipe for the determination of the tangential velocity of that projectile. In this case the distance GJ can be used as the measure of the tangential velocity and the distance SU located in the SURF Circle describes the arm for the evaluation of the moment of tangent momentum- see Figure 7.

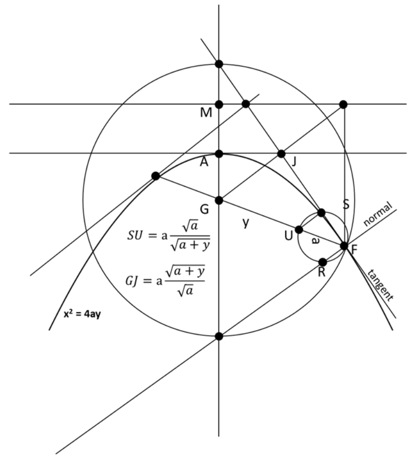

Figure 7. MAG Parabola with the distance GJ (this length represents the tangential velocity of the projectile) and with the SURF Circle (Ptolemy's Circle) where the distance SU represents the arm for the evaluation of the moment of tangent momentum

We were inspired by W.R. Hamilton who in 1847 discovered his concept for the Kepler's ellipse that is known as the hodograph. This approach was several times forgotten and its Beauty was several times rediscovered by many researchers. E.g., Richard Feynman in his "Lost lecture" made this concept very well known for our generation.

Another approach can be taken from the Great Old Master-Gottfried Wilhelm Leibniz. Gottfried Wilhelm Leibniz during his preparation work on the infinitesimal calculus introduced six parameters for a point on the parabola: abscissa, ordinate, length of tangent, subtangent, length of normal, and subnormal. Leibniz studied the change of these six parameters and their dependence on the direction of tangent and normal and their dependence on the curvature in that point on the parabola. For his infinitesimal triangle Leibniz used the ratio of ordinate to subtangent and the ratio of subnormal to ordinate. Can we extract any additional information hidden in those six Leibniz's parameters? For details see D. Dennis (1995), D. Dennis and J. Confrey (1995). [See also A.R. Hall (2015) and T. Sonar (2018)].

For the tangent velocity v_T of a projectile on the parabolic path we propose to use the ratio of length of the normal to the subnormal:

$$v_T = v_{ESC}\frac{\sqrt{a+y}}{\sqrt{a}} = v_{ESC}\frac{2\sqrt{a}\sqrt{a+y}}{2a} \tag{3}$$

where v_{ESC} is the very well-known escape velocity just to leave the Galileo's empty focus-it is the horizontal component of the projectile velocity at the beginning. The first part of this Equation was given to us by W.R. Hamilton, the second part of that Equation was presented to us by G. W. Leibniz.

Now, we can test the validity of the Newton's formula expressed in the trigonometric language and compare these trigonometric formulae with formulae obtained in other mathematical languages.

The famous Newton's formula can be trigonometrically expressed as:

$$\begin{aligned} F &= \frac{GMm}{R^2} = m\,g = \\ &= m\,\frac{v_T^2}{\rho}\frac{1}{\cos\alpha} = m\,\frac{v_{esc}^2\left(\frac{a+y}{a}\right)}{\frac{2\left(a+y\right)^{3/2}}{\sqrt{a}}}\frac{\sqrt{a+y}}{\sqrt{a}} = \\ &= \frac{m\,v_{esc}^2}{2\,a} = \frac{m\,v_c^2}{a} \end{aligned} \tag{4}$$

We have inserted the expression for the escape velocity $v_{ESC} = 2^{½}\, v_C$, where v_C is the velocity for the circular orbit. At the end we have obtained the Newton-Huygens formula describing the motion of projectile on circular path around the Galileo's empty focus.

7. Galileo's Parabola Observed from the Contrapedal Curve with the Pedal Point in the Galileo's Empty Focus and from the Ptolemy's Circle (Hodograph)

We want to find an expression for the normal velocity v_N of a projectile on the parabolic path. In this case the MAG Parabola shows Her original Feature and Beauty. We will use the contrapedal distance GE for the arm of the moment of the normal momentum from Figure 8.

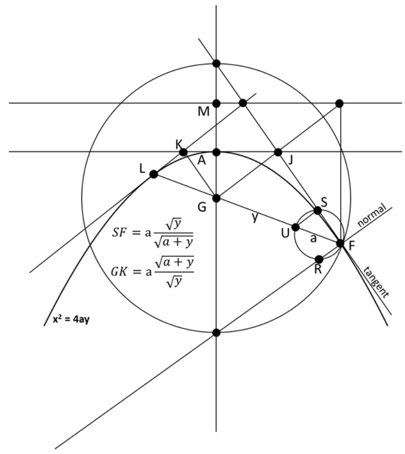

Figure 8. MAG Parabola with the distance GK (length of the arm for the moment of normal momentum) and with the SURF Circle (Ptolemy's Circle-the Hodograph) where SF represents the normal velocity on the parabolic path

For the normal velocity v_N of a projectile on the parabolic path we propose to use the ratio of the abscissa to the length of the normal:

$$v_N = v_{esc}\frac{2\sqrt{ay}}{2\sqrt{a}\sqrt{a+y}} = v_{esc}\frac{\sqrt{y}}{\sqrt{a+y}} \tag{5}$$

where v_{ESC} is the very well-known escape velocity of an object just to leave the Galileo's empty focus.

8. Moment of the Tangent Momentum and the Moment of the Normal Momentum of the Galileo's Parabola

Based on the formulae in Tables I and in Figures in this contribution we can evaluate the moment of the tangent momentum L_T and to introduce a new physical quantity-the moment of the normal momentum L_N.

The moment of momentum L is defined as the product of the linear momentum with the length of the moment arm, a line dropped perpendicularly from the origin onto the path of the particle. It is this definition: L = (length of moment arm) x (linear momentum).

The moment of the tangent momentum L_T for the Galileo's parabola is given as:

$$L_T = m\ v_T SU = m\ v_{esc}\frac{\sqrt{a+y}}{\sqrt{a}}a\ \frac{\sqrt{a}}{\sqrt{a+y}} = m\ v_{esc}a = \sqrt{2}\ m\ v_c a \tag{6}$$

where m is the mass of that projectile, v_T the tangent velocity of that projectile on the parabolic path and SU is the length of the moment arm (the distance in the Ptolemy's Circle). The moment of the tangent momentum L_T is constant during the complete parabolic path of the Galileo's parabola. Therefore, there is no contribution to the torque from this moment of the tangent momentum. This is very well-known experimental fact documented in the existing literature.

The moment of the normal momentum L_N for the Galileo's parabola is given as:

$$L_N = m\ v_N GK = m\ v_{esc}\frac{\sqrt{y}}{\sqrt{a+y}}a\ \frac{\sqrt{a+y}}{\sqrt{y}} = m\ v_{esc}a = \sqrt{2}\ m\ v_C a \tag{7}$$

where m is the mass of that projectile, v_N the normal velocity of that projectile on the parabolic path and GK is the length of the moment arm (the distance between the Galileo's empty focus and the tangent to the point L). The moment of the normal momentum is constant during the complete path of the Galileo's parabola. Therefore, there is no contribution to the torque from this moment of the normal momentum. This is very well-known experimental fact documented in the existing literature.

9. Longitudinal and Perpendicular Total Time of the Projectile Flight on the MAG Parabola

The total time of the projectile flight can be expressed both for the longitudinal and perpendicular motion and we will get the same expression. The perpendicular time T_P can be expressed as:

$$T_P = 2\,\frac{v_T \sin\theta}{g} = 2\sqrt{\frac{2\,y}{g}} \tag{8}$$

The longitudinal time T_L can be expressed as:

$$T_L = \frac{4\sqrt{a\,y}}{v_T \cos\theta} = 2\sqrt{\frac{2\,y}{g}} \tag{9}$$

We can measure time in both directions without the introduction of the concept of the "imaginary time".

10. Six Expressions for the Projectile Speed on the MAG Parabola-The "Rashomon Effect"

We can express the projectile speed on the MAG Parabola in six possible formulae-see Table 3.

Table 3. Six expressions for the projectile speed on the MAG Parabola

Instantaneous longitudinal speed
$v_x = v_T \cos\alpha = v_{esc}$
Instantaneous perpendicular speed
$v_y = v_T \sin\alpha - g\,t = v_{esc}\frac{\sqrt{y}}{\sqrt{a}}$
Instantaneous tangential speed
$v_T = v_{esc}\frac{\sqrt{a+y}}{\sqrt{a}}$
Instantaneous normal speed
$v_N = v_{esc}\frac{\sqrt{y}}{\sqrt{a+y}}$
Average perpendicular speed
$v_P^{AV} = \frac{v_T \sin\alpha}{2} = \frac{v_{esc}\frac{\sqrt{y}}{\sqrt{a}}}{2}$
Average parabolic path speed
$v_{par}^{AV} = \frac{v_{esc}}{2}\left[\frac{\sqrt{a+y}}{\sqrt{a}} + \frac{\sqrt{a}}{\sqrt{y}}\operatorname{arcsin}h\frac{\sqrt{y}}{\sqrt{a}}\right]$

We might describe the projectile motion from different angles and we will get those six expressions-this is a kind of the famous "Rashomon effect"-each observer will communicate a different description of the same event.

11. Dürer-Simon's Parabola and the Tangential and Normal Accelerations of the Projectile

We were inspired by the Albrecht Dürer's picture of parabola (1515) depicting the reflective properties of parabola that might describe the direction of the total acceleration towards the Galileo's empty focus when the projectile approaches the Galileo's empty focus. On the same day (24.02.2019) we have found the very stimulating concept of Raul A. Simon (2017) that we have re-written as:

$$g^2 = a_T^2 + a_N^2 = g^2\left(\sin\alpha\right)^2 + g^2\left(\cos\alpha\right)^2 = g^2\frac{y}{a+y} + g^2\frac{a}{a+y} \tag{10}$$

In this Dürer-Simon's Parabola the total acceleration g = 9.81 $msec^{-2}$ is directing towards to the Galileo's empty focus when the projectile approaches the vertex of parabola. Behind the vertex the total acceleration of the projectile-g = 9.81 $msec^{-2}$-directs towards the second focus of Galileo's Parabola located in the "infinity" in the center of the Earth. The changing direction of the total acceleration results from the joint interplay of the Earth's and projectile's gravitons and their reflections and refractions inside of the moving projectile. It could be interesting to experimentally evaluate both the tangential and normal acceleration of the moving projectile on the Galileo's Parabola. (In the huge literature describing properties of the Galileo's Parabola we could not find a paper similar to the Simon's concept. The further discussion to this topic we will leave to Readers of this Journal).

Figures 9 and 10 show the direction of the total acceleration g towards to the Galileo's empty focus-G-for the projectile approaching to the vertex-A-of that Parabola. Behind the vertex-A-of that Parabola the total acceleration g of the projectile directs towards the second focus of that Parabola in the "infinity"-JS-that is occupied by the Center of the Earth.

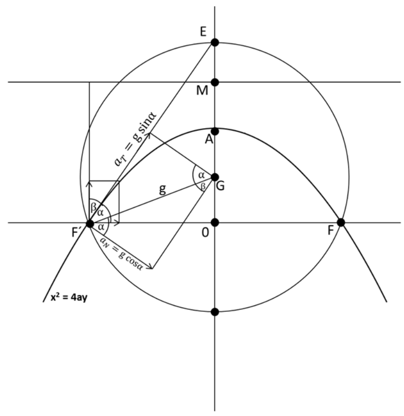

Figure 9. Dürer-Simon's Parabola with tangential and normal accelerations of the projectile and the total acceleration g directing towards the Galileo's empty focus G

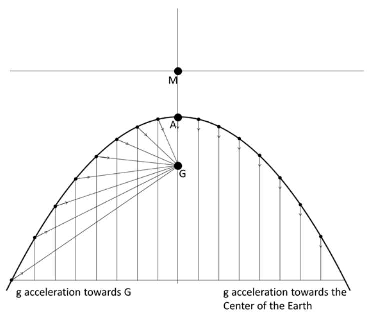

Figure 10. Dürer-Simon's Parabola with the total acceleration g directing towards the Galileo's empty focus G and towards the second focus JS occupied by the Center of the Earth

12. The "Vis Viva" Controversy between the Newtonian and Leibnizian Schools

The "vis viva" controversy played a very important role between two great Schools-the Newtonian and Leibnizian Schools. Based on our research we propose to solve this dilemma satisfying both sides of this long discussion.

We propose to use the expression for the kinetic energy E = ½ mv^2 for events on parabolic orbits and paths.

We propose to use the expression for the "vis viva" E = mv^2 for events on hyperbolic orbits and paths.

E.g., if we will insert the "vis viva" expression into the Soldner's formula (1801) (e.g., Fengyi Huang in 2017), then we will get the right experimental value for the deflection of light on hyperbolic orbits.

13. "Antikythera Mechanism" behind the MAG Parabola

We propose to use the very-well known Antikythera Mechanism as an analogy for the visible MAG Parabola-a part of our Aristotelian World-connected deeply with invisible curves from the Plato's Realm-Pappus' Directrix, Apollonius' auxiliary Circle, Newton's Evolute, Leibniz's six parameters (abscissa, ordinate, length of tangent, subtangent, length of normal, and subnormal), Ptolemy's Circle (Hodograph), and Dürer-Simon's Parabola.

Are there some more hidden curves in the Plato's Realm connected to the MAG Parabola? How to distinguish the real physical meaning written in those curves from fictious events if both are mathematically correct? How to work with the mathematical camouflage used by Nature to protect Her Secrets?

The detailed analysis of these ideas we will leave to the Readers of this Journal better educated in mathematics.

14. Conclusions

1. We have presented some quantitative properties of the Galileo's parabola in Tables I and II and Figures 1-10.

2. We have studied the interplay of the directrix of parabola **M** (=Menaechmus-the discoverer of parabola) with the vertex of parabola **A** (=Apollonius of Perga) and the occupied focus of parabola **N** (=Newton) in the **MAG** Parabola.

3. We have discovered a new trigonometric road leading to the Newton's gravitational formula for the Galileo's Parabola.

4. We have used the pedal curve with the pedal point in the Galileo's empty focus to determine the expression for the tangent velocity.

4. We have employed Leibniz's length of the normal and subnormal to get an expression for the tangent velocity of an object on the Galileo's parabolic path.

5. We have employed Leibniz's abscissa and the length of the normal to get an expression for the normal velocity of an object on the Galileo's parabolic path.

6. We have derived an expression for the moment of tangent momentum of an object on the Galileo's parabolic path.

7. We have derived an expression for the moment of normal momentum of an object on the Galileo's parabolic path.

8. We have used the Ptolemy's Circle as the new Hodograph for the Galileo's parabolic path.

9. We have used the Dürer-Simon's Parabola to depict the tangential and normal accelerations of the projectile on the Galileo's parabolic path.

10. We have presented a possible solution of the "vis viva controversy" satisfying both the Newtonian and Leibnizian Schools.

11. Are there some more hidden curves in the Plato's Realm connected to the Galileo's Parabola? How to distinguish the real physical meaning written in those curves from fictious events if both are mathematically correct? How to work with the mathematical camouflage used by Nature to protect Her Secrets?

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Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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