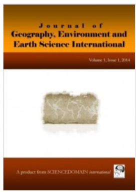

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An Analytic Solution of Mathematical Model of Boussinq's Equation in Homogeneous Porous Media During Infiltration of Groundwater Flow

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The present paper discusses with the problem of groundwater given by Boussinesq's equation infiltration phenomenon in unsaturated porous media. The governing nonlinear partial differential equation is solved using optimal homotopy analysis method. The solution gives a height of free surface of infiltrated water table in unsaturated homogeneous porous media (soil) by using appropriate initial guess value of the solution. The numerical as well as graphical representation of the solution is given by Mathematica coding.

Keywords: Boussinesq's equation; unsaturated porous media; optimal homotopy analysis method; convergence control parameter; groundwater flow.

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1 INTRODUCTION

The groundwater flow plays an important role in various fields like fluid dynamics, agriculture, environmental problems, chemical engineering, nuclear waste disposal problems and bio mathematics. The infiltration phenomenon is useful to agriculture purpose, contamination of water and to control salinity of water. Such types of problems are also useful to measure moisture content of water in vertical one-dimensional ground water recharge. A confined aquifer is aquifer bounded from above and from below by impervious formations, while unconfined aquifer is bounded above by water table. Infiltration is process in that groundwater moves through the pores of an unsaturated porous medium (soil). It is governed by mainly two forces, (1) gravity and (2) capillary action. The present model was developed first by Boussinesq [1] in 1903 and is related to a original motivation of Darcy [2]. The present mathematical model deals with the filtration of an incompressible fluid (typically water) through a porous stratum, the main problem in groundwater infiltration. According to Polubarinova-Kochina [3] and Scheidegger [4] the flow infiltrated water enters in unsaturated soil, the infiltered water will start developing a curve between saturated porous medium and unsaturated porous medium of unconfined aquifer known as water table. This height of the free surface of water table is investigated in the mathematical model of Boussinesq's equation.

The Homotopy analysis method (HAM)[5, 6] is based on the concept of homotopy, a fundamental concept in topology and differential geometry, which was first time introduced by Shijun Liao in his PhD thesis (1992). The optimal homotopy analysis method is a combination of HAM and the optimization of convergence control parameter c_0. There are many different method based on HAM like optimal homotopy asymptotic method [7, 8, 9, 10]. The HAM is a general analytic approach to get series solutions of various types of non-linear equations, including algebraic equations, ordinary differential equations, partial differential equations, deferential-integral equations, and coupled equations of them. The optimal homotopy analysis method based on homotopy analysis method has a great advantage that, in general its validity does not depend upon on small or large parameters, and it is easy to adjust the convergence region and rate of approximation series by taking optimal value of convergence control parameter. Therefore, the optimal homotopy analysis method handles linear and nonlinear problems without any assumption and restriction.

The present paper discusses the approximate analytical solution of nonlinear partial differential equation for an infiltration phenomenon, to examine the height of the free surface of the water table in unsaturated homogeneous soil. The changes in saturated porous media and distribution of pore pressures can be calculated by using the optimal homotopy analysis method.

2 MATHEMATICAL MODEL OF BOUSSINQ'S EQUATION

Consider reservoir field with water of height $h_{\max}$ in unconfined aquifer and surrounding of this reservoir is unsaturated homogeneous soil. Infiltration phenomenon is well demonstrated in the following figure 1 that shows a vertical cross section of the reservoir.

The infiltration is a process in which the groundwater of the reservoir has entered into the unsaturated soil through vertical permeable wall. The infiltrated water will enter in unsaturated soil then the infiltered water will develop a curve between saturated homogeneous soil and unsaturated homogeneous soil, which is called water table. To measure the height of the free surface of a water table is the basic purpose of the investigation.

The governing equation for the height of infiltered water is obtained in the form of a non-linear partial differential equation known as Boussinesq's equation. The atmospheric pressure in a dry region by using relation between pressure and height of free surface and velocity of infiltered water can be calculated by Darcy's law. To develop the mathematical modeling of the infiltration phenomenon consider the following certain assumptions.

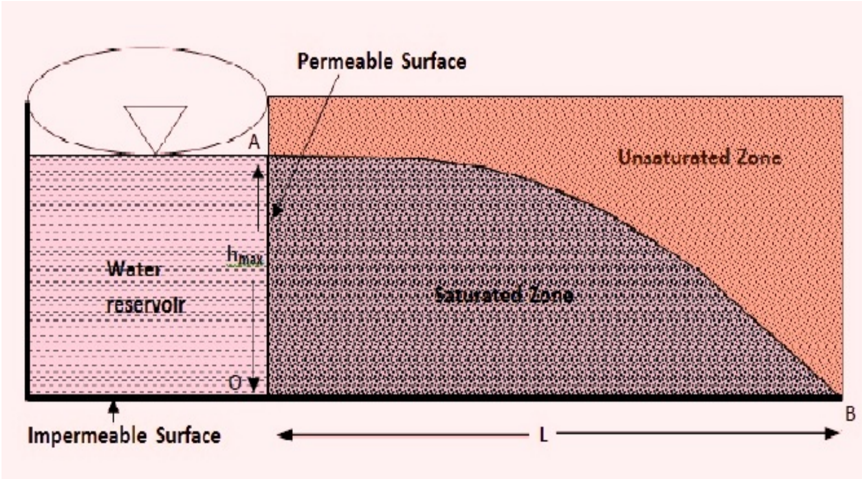

Figure 1: The schema of ground infiltration

Table 1: Numerical value for the height of infiltrated groundwater $H(X,T)$

X	T=0.0	T=0.1	T=0.2	T=0.3	T=0.4	T=0.5	T=0.6	T=0.7	T=0.8	T=0.9	T=1.0
0.0	09873	0.98894	0.99055	0.99217	0.99379	0.9954	0.99704	0.99867	1.0003	1.0019	1.0035
0.1	0.9873	0.9865	0.9857	0.9849	0.9841	0.9833	0.9824	0.9816	0.9807	0.9799	0.9791
0.2	0.9871	0.9834	0.9796	0.9758	0.9720	0.9682	0.964	0.9605	0.9566	0.9527	0.9487
0.3	0.9867	0.9794	0.9721	0.9647	0.9572	0.9498	0.9422	0.9346	0.9270	0.9193	0.9112
0.4	0.9861	0.9744	0.9628	0.9510	0.9392	0.9273	0.9153	0.9032	0.8910	0.8788	0.8665
0.5	0.9851	0.9684	0.9515	0.9346	0.9175	0.9002	0.8828	0.8653	0.8476	0.8298	0.8118
0.6	0.9837	0.9609	0.9379	0.9148	0.8913	0.8677	0.8438	0.8197	0.7953	0.7708	0.7459
0.7	0.9818	0.9519	0.9216	0.8910	0.8601	0.8288	0.7971	0.7651	0.7327	0.6999	0.6667
0.8	0.9794	0.9410	0.9022	0.8628	0.8229	0.7825	0.7414	0.6999	0.6577	0.6149	0.5716
0.9	0.9763	0.9281	0.8796	0.8293	0.7787	0.7274	0.6752	0.6221	0.5681	0.5133	0.4575
1.0	0.9725	0.9127	0.8517	0.7897	0.7264	0.6620	0.5964	0.5294	0.4611	0.3916	0.3206

1. The stratum has height H and lies on top of a horizontal impervious bed, which we label as $z = 0$.
2. Ignore the transversal variable y and
3. The water mass which infiltrates the soil occupies a region described as $\Omega = \{(x,\ z) \in R : z \leq h(x,\ t)\}$

In practical terms, it is assumed that there is no region of partial saturation. This is an evolution model. Clearly, $0 \leq h\,(x,t) \leq h_{\max}$ where, $h_{\max}$ is the maximum height and the free boundary surface, $h\,(x,t)$ is also an unknown variable of this mathematical model. For the sake of simplicity and for the practical computation after introducing suitable assumptions, the hypothesis of almost horizontal flow, it is assumed that the flow has almost horizontal speed. Here, the $y-$ component of the velocity of infiltered water will be zero. Here, $u >> (u, 0)$ so that has small gradients. It follows $h(x,t)$ that in the vertical component.

$$\rho\left(\frac{\partial u_z}{\partial t} + u \cdot \nabla u_z\right) = -\frac{\partial p}{\partial z} - \rho g. \qquad (2.1)$$

We may neglect the inertial term. Integration of an equation (2.1) with respect to z gives for this first approximation $p + \rho g z =$ constant. Now we calculate the constant on the free surface $z = h\,(x,\ t)$. If we impose continuity of the pressure across the interface, we have $p = 0$ assuming constant atmospheric pressure in the air that fills the pores of the dry region $z > h\left(x,\ t\right)$, we get

$$p = \rho g(h - z). \quad (2.2)$$

In other words, the pressure is determined by means of the hydrostatic approximation. We go now to the mass conservation law, which will give us the equation. We proceed as follows: we take a section $S = (x,\ x + a) \times (0,\ C)$ then

$$\varepsilon \frac{\partial}{\partial t} \int_{x}^{x+a} \int_{0}^{h} dydx = -\int_{\partial s} u \cdot n\, dl, \quad (2.3)$$

where ε is the porosity of the medium, i.e., the fraction of volume available for the flow circulation, and u is the velocity, which obeys Darcy's law in the form that includes gravity effects

$$u = -\frac{k}{\mu} \nabla \left(p + \rho g z\right). \quad (2.4)$$

On the, right-hand lateral surface we have $u \cdot n \approx (u, 0) \cdot (1, 0) = u$ i.e. $-\left(\frac{k}{\mu}\right) p_x$ while on the left-hand side we have $-u$. Using the formula for p and differentiating with respect to x, we get

$$\varepsilon \frac{\partial h}{\partial t} = \frac{\rho g k}{\mu} \int_{0}^{h} \frac{\partial}{\partial x} h\, dz. \quad (2.5)$$

Thus, we obtained Boussinesq's equation as

$$\frac{\partial h}{\partial t} = \frac{\rho g k}{2\mu\varepsilon} \frac{\partial^2}{\partial x^2} h^2, \quad (2.6)$$

with an initial and boundary condition as

$$h\left(x, 0\right) = h\left(x\right) \quad ; \ t = 0,\ x > 0$$

$$h\left(0, t\right) = h_{\max} \quad ; \ x = 0,\ t > 0.$$

To generalize the model, choosing new dimensionless variables as

$$H = \frac{h}{L},\ X = \frac{x}{L}\ \ and\ \ T = \frac{\rho g k}{L \mu \varepsilon} t.$$

Then, equation (2.6) is in the following form

$$\frac{\partial H}{\partial T} = H \frac{\partial^2 H}{\partial X^2} + \left(\frac{\partial H}{\partial X}\right)^2, \quad (2.7)$$

with initial and boundary conditions as

$$\begin{array}{ll} H\left(X, 0\right) = e^{-X} & ;\ T = 0,\ X > 0 \\ H\left(0, T\right) = H_{\max} & ;\ X = 0,\ T > 0. \end{array} \quad (2.8)$$

3 SOLUTION OF MATHEMATICAL MODEL BY OHAM

The partial differential equation (2.7), which arise in groundwater flow describe the height of infiltrate water. In order to apply OHAM [5, 11] we choose the initial guess [12] and the auxiliary linear operator [13] as follow

$$H_0\left(X, T\right) = \left(H_{\max} - \frac{X\ T\ e^X}{4}\right) \ \ \&\ \ \mathcal{L}\left[H\left(X, T; q\right)\right] = \frac{\partial H\left(X, T; q\right)}{\partial X}, \quad (3.1)$$

which posses the property $\mathcal{L}\left(c_1\right) = 0;\ c_1$ is an integral constant to be determine by initial condition. The zero-order deformation equation is given as

$$\left(1 - q\right) \mathcal{L}\left[\phi\left(X, T; q\right) - H_0\left(X, T\right)\right] = c_0 q \mathcal{N}\left[\phi\left(X, T; q\right)\right], \quad (3.2)$$

where c_0 is convergence control parameter and $q \in [0, 1]$ is embedding parameter. The corresponding m^{th}-order deformation equation is

$$\mathcal{L}\left[H_m\left(X, T\right) - \chi_m H_{m-1}\left(X, T\right)\right] = c_0 \delta_m\left[H_{m-1}\left(X, T\right)\right], \quad (3.3)$$

where

$$\delta_m\left(H_{m-1}\right) = \left(H_{m-1}\right)_T - \sum_{r=0}^{m-1} H_r \left(H_{r-m-1}\right)_{XX} - \sum_{r=0}^{m-1} \left(H_r\right)_X \left(H_{r-m-1}\right)_X.$$

Applying the inverse operator, we have

$$H_m\left(X, T\right) = \chi_m H_{m-1}\left(X, T\right) + c_0 \mathcal{L}^{-1}\left[\delta_m\left[H_{m-1}\left(X, T\right)\right]\right], \quad (3.4)$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Consequently we get the approximations of $H\left(X,T\right)$ as

$$H_1\left(X,T\right) = (0.25\mathrm{e}^X + 0.243\mathrm{e}^X T - 0.25\mathrm{e}^X X + 0.243\mathrm{e}^X TX - 0.063\mathrm{e}^{2X}T^2X - 0.063\mathrm{e}^{2X}T^2X^2)c_0$$

$$\begin{aligned} H_2\left(X,T\right) = & (0.25\mathrm{e}^X + 0.243\mathrm{e}^X T - 0.25\mathrm{e}^X + 0.243\mathrm{e}^X TX - 0.063\mathrm{e}^{2X}T^2X \\ & - 0.063\mathrm{e}^{2X}T^2X^2)c_0 + c_0(-0.47\mathrm{e}^X Tc_0 + 0.063\mathrm{e}^{2X}Tc_0 + 0.121\mathrm{e}^{2X}T^2c_0 \\ & + 0.485\mathrm{e}^X Xc_0 - 0.235\mathrm{e}^X TXc_0 + 0.485\mathrm{e}^{2X}T^2Xc_0 - 0.031\mathrm{e}^{3X}T^3Xc_0 \\ & - 0.186\mathrm{e}^{2X}TX^2c_0 + 0.242\mathrm{e}^{2X}T^2X^2c_0 - 0.094\mathrm{e}^{3X}T^3X^2c_0 - 0.047\mathrm{e}^{3X}T^3X^3c_0) \end{aligned}.$$

and so on.

Using Yabushita's approach [14] for finding the square residual errors at different order of approximation as

$$E_m\left(c_0\right) = \iint\limits_{\Omega} \left[\mathcal{N}\left\{\sum_{n=0}^{m} H_n\left(x,t\right)\right\}\right]^2 d\Omega. \tag{3.5}$$

In a real world problem the double integration of sum of square residual is very difficult, so we use it's approximate sum form as

$$E_m\left(c_0\right) = \frac{1}{\left(M+1\right)\left(N+1\right)} \sum_{i=0}^{M}\sum_{j=0}^{N} \left\{\mathcal{N}\left[\sum_{n=0}^{m} H_n\left(\frac{i}{M},\frac{j}{N}\right)\right]^2\right\}. \tag{3.6}$$

We calculate equation (3.6) for $M = 50$ & $N = 50$ numbers of points and we get the optimal value of convergence control parameter $c_0 = 0.01157372330371058$ with minimum square residual error $E_6 = 1.1047E - 01$, which can be notice by Figure 2.

Using initial guess value and the sixth order approximation, we get the approximate solution of the governing partial differential equation (2.7) as

$$H\left(X,T\right) = \left\{ \begin{array}{l} (0.25\mathrm{e}^X + 0.243\mathrm{e}^X T - 0.25\mathrm{e}^X X + 0.243\mathrm{e}^X TX - 0.063\mathrm{e}^{2X}T^2X \\ -0.063\mathrm{e}^{2X}T^2X^2)0.012 + (0.25\mathrm{e}^X + 0.243\mathrm{e}^X T - 0.25\mathrm{e}^X + 0.243\mathrm{e}^X TX \\ -0.063\mathrm{e}^{2X}T^2X - 0.063\mathrm{e}^{2X}T^2X^2)0.012 + 0.012(-0.47\mathrm{e}^X T0.012 \\ + 0.063\mathrm{e}^{2X}T0.012 + 0.121\mathrm{e}^{2X}T^2 0.012 + 0.485\mathrm{e}^X X0.012 \\ -0.235\mathrm{e}^X TXc_0 + 0.485\mathrm{e}^{2X}T^2X0.012 \; - 0.031\mathrm{e}^{3X}T^3X0.012 \\ - 0.186\mathrm{e}^{2X}TX^2 0.012 + 0.242\mathrm{e}^{2X}T^2X^2 0.012 \\ - 0.094\mathrm{e}^{3X}T^3X^2 0.012 - 0.047\mathrm{e}^{3X}T^3X^3 0.012) + ... \end{array} \right.$$

We can prove the analytical convergence of OHAM solutions. To do this, let us state the following theorem which gives sufficient conditions for the convergence or divergence of the homotopy series.

Theorem 1. Suppose that $A \subset R$ be a Banach space donated with the L^2 norm, over which the sequence $U_k\left(x,t\right)$ of the homotopy series $U\left(x,t;q\right) = \sum\limits_{k=1}^{\infty} U_k\left(x,t\right)q^k$ is defined for a prescribed value of c_0. Assume also that the initial approximation $U_0\left(x,t\right)$ remains inside the disc of the solution $U\left(x,t\right)$. Taking $r \in R$ be a constant, the following statements hold true:

1. If $\left\|U_{k+1}\left(x,t\right)\right\| \leq r\left\|U_k\left(x,t\right)\right\|$ for all k, given some $0 < r < 1$, then the series solution converges absolutely at $q = 1$ over the domain of definition of $\left(x,t\right)$.
2. If $\left\|U_{k+1}\left(x,t\right)\right\| \geq r\left\|U_k\left(x,t\right)\right\|$ for all k, given some $r > 1$, then the series solution diverges at $q = 1$ over the domain of definition of $\left(x,t\right)$.

Proof. (1) Let $S_n\left(x,t\right)$ denote the sequence of partial sum of the homotopy series, we need to show that $S_n\left(x,t\right)$ is a Cauchy sequence in A. Also we have

$$\left\|S_{n+1}\left(x,t\right) - S_n\left(x,t\right)\right\| = \left\|U_{n+1}\left(x,t\right)\right\| \leq r\left\|U_n\left(x,t\right)\right\| \leq r^2\left\|U_{n-1}\left(x,t\right)\right\| \leq ... \leq r^{n+1}\left\|U_0\left(x,t\right)\right\|. \tag{3.7}$$

It should be remarked that owing to (3.7), all the approximations produced by the homotopy method will lie within the disk of $S_n(x,t)$. For every $m, n \in \mathbb{N}; n \geq m$, also using equation (3.7) and the triangle inequality, we have

$$\|S_n - S_m\| = \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + ... + (S_{m+1} - S_m)\| \leq \frac{1 - r^{n-r}}{1 - r} r^{m+1} \|U_0\|. \quad (3.8)$$

Since $0 < r < 1$ we get from equation (3.8)

$$\lim_{n,m \to \infty} \|S_n(x,t) - S_m(x,t)\| = 0. \quad (3.9)$$

Therefore, $S_n(x,t)$ is a Cauchy sequence in the Banach space A, and we know that all Cauchy sequence in Banach space are convergent, that is the series solution is convergent.

The proof of (2) follows from the fact that under the hypothesis supplied in (2), suppose if possible there exist a number l such that, $l > r > 1$, so that the interval of convergence of the power series is $| q |< 1/l < 1$, which obviously contradiction with $q = 1$. □

4 NUMERICAL AND GRAPHICAL REPRESENTATION

Using the optimal value of convergence control parameter, Numerical and graphical presentations of a equation (2.7) have been obtained by MATHEMATICA software. Figure 3 shows the graph of height H vs X for time $T = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ is decreasing as distance as well as time increasing, which is consistent with a physical situation.

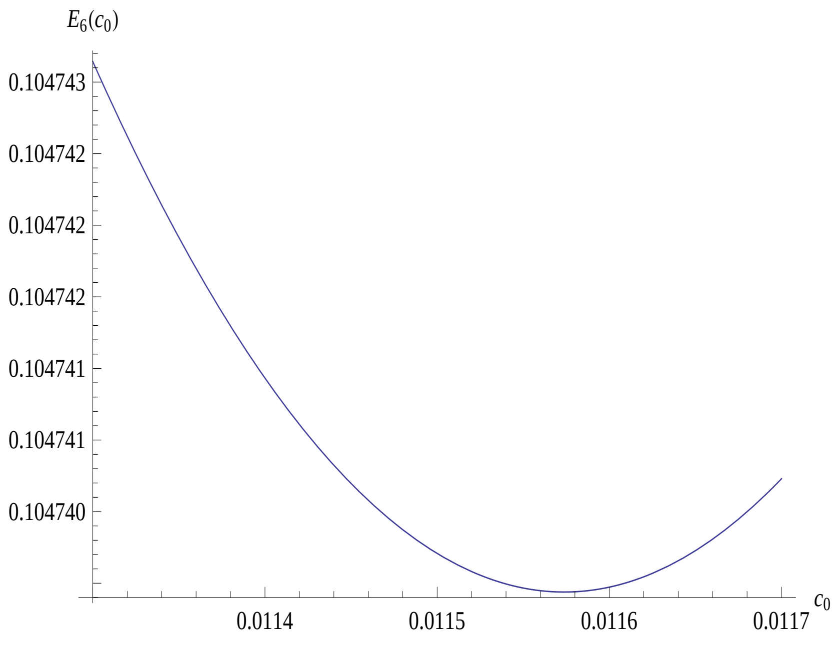

Figure 2: Square residual error at 6^{th}-order approximation

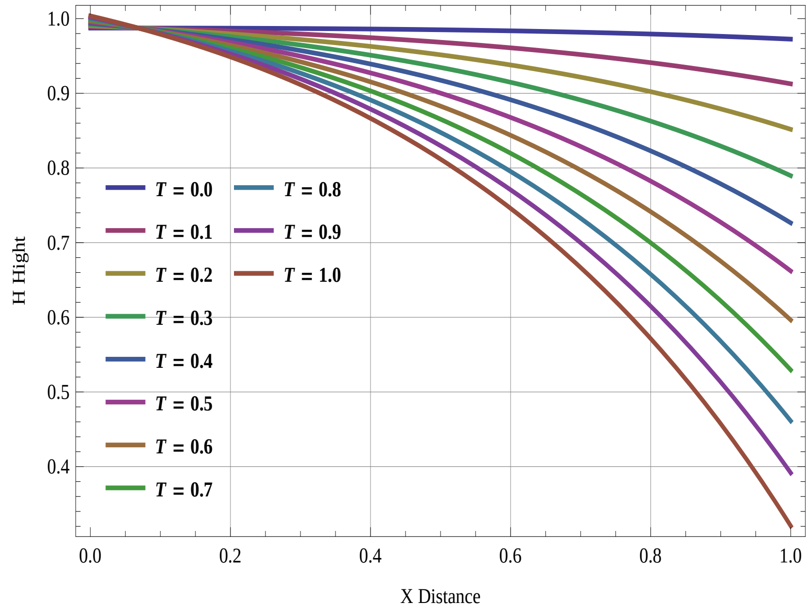

Figure 3: Height of Infiltrated Groundwater $H(X,T)$ at different distance with a fix time level $T = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ & $H_{max} = 0.97$.

5 CONCLUSIONS

The equation (2.7) represents a height of infiltrated groundwater for any distance X and time $T > 0$. Using optimal homotopy analysis method, the optimal value of convergence control parameter $c_0 = 0.011573723303371058$ is successfully obtained by minimizing the square residual error. Figure 3 represents the height H for infiltrated groundwater vs. distance X and time T; it shows that height of infiltrated groundwater is decreasing as distance X increasing for fix time level T. From figure 3, it can be concluded that for infiltration of groundwater is decreasing as distance X increasing and when time T increasing. Therefore, the solution of one-dimensional Boussinq's equation by optimal homotopy analysis method is graphically as well as physically consistent. Finally, we can concluded that the infiltration rate of groundwater is decreasing with distance X, as well as time T, increases using optimal homotopy analysis method.

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COMPETING INTERESTS

The authors declare that no competing interests exist.

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