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Universal Walsh Series with Monotone Coefficients in Weighted L^p Spaces

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Abstract

For any $0 < \delta < 1$ there exists a measurable function $\mu(x)$, $0 < \mu(x) \leq 1$, with $|\{x \in [0,1]; \mu(x) \neq 1\}| < \delta$, and a series in the Walsh system $\{\varphi_n\}$ of the form

$$\sum_{n=0}^{\infty} a_n \varphi_n, \quad \text{with} \quad |a_n| \searrow 0,$$

such that for any $p \geq 1$ and any function $f \in L^p_\mu(0,1)$ one can fined subseries of above series converging to f in $L^p_\mu(0,1)$.

Keywords: Walsh system, convergence, weighted spaces.

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1 Introduction

Let $\mu(x)-$ia a weighted function and let

$$L^p_\mu(0,1) = \{f; \int_0^1 |f(x)|^p \mu(x)dx < \infty\}$$

A system of functions

$$\{f_k(x)\}_{k=0}^{\infty}, \quad f_k(x) \in L^1_\mu[0,1]$$

is called a system of representation for weighted $L^1_\mu[0,1]$ class, if for any $f(x) \in L^1_\mu[0,1]$ there is a series $\sum_{k=1}^{\infty} a_k f_k(x)$ which converges to the $f(x)$ in the metric $L^1_\mu[0,1]$.

Note, that many papers are devoted (see [1]- [14]) to the question on existence of various types of representation by different systems in the sense of convergence almost everywhere, on a measure, in L^p metric.

In this paper we prove the following theorem:

Theorem *For any $0 < \delta < 1$ there exists a measurable function $\mu(x),\ 0 < \mu(x) \leq 1,$ with $|\ \{x \in [0,1]; \mu(x) \neq 1\}\ | < \delta$, and a series in the Walsh system $\{\varphi_n\}$ of the form*

$$\sum_{n=1}^{\infty} a_n \varphi_n, \quad \textit{with} \quad |a_n| \searrow 0,$$

such that for any $p \geq 1$ and any function$f \in L^p_\mu(0,1)$ there exists a subseries

$$\sum_{k=1}^{\infty} a_{n_k} \varphi_{n_k}$$

converging to f in $L^p_\mu(0,1)$.

Recall the following definition: a series $\sum_{n=1}^{\infty} a_n \varphi_n$ is said to be universal with respect to subseries in the space $L^p_\mu(0,1)$, where $p \geq 1$ is fixed, if for each function $f(x) \in L^p_\mu(0,1)$, one can select a subseries $\sum_{k=1}^{\infty} a_{n_k} \varphi_{n_k}$ which converges to $f(x)$ in $L^p_\mu(0,1)$ norm .

Note that in this theorem it is impossible to replace $L^p_\mu(0,1)$ with $L^p(0,1)$. This is obvious: for instance if for the function $(|a_1| + 1)\varphi_1(x)$ there exists subseries $\sum_{k=1}^{\infty} a_{n_k} \varphi_{n_k}$, ($n_k \nearrow$) of series $\sum_{n=1}^{\infty} a_n \varphi_n(x)$ which converges to this function by $L^p(0,1)$ norm, then it follows that $a_{n_k} = \int_0^1 (|a_1| + 1)\varphi_1(x)\varphi_{n_k}(x)dx$ for all $k \geq 1$, hence, if $n_1 > 1$ we get $a_{n_k} = 0$ for all $k \geq 1$, else if $n_1 = 1$ we get $1 + |a_1| = a_1$, which is contradiction.

The following problems remain open:

Question. *Is this theorem true for the trigonometric system?*

2 Proof of Main Lemmas and Theorem

The Walsh system, an extension of the Rademacher system, may be obtained in the following manner:

Let r be the periodic function, of least period 1, defined on $[0, 1)$ by

$$r = \chi_{[0,1/2)} - \chi_{[1/2,1)}.$$

The Rademacher system, $R = r_n : n = 0, 1, ...$, is defined by the conditions

$$r_n(x) = r(2^n x),\ \forall x \in R, n = 0, 1, ...,$$

and, in the ordering employed by Paley (see [15] and [16]), the n-th element of the Walsh system $\{\varphi_n\}$ is given by

$$\varphi_n = \prod_{k=0}^{\infty} r_k^{n_k}, \tag{1}$$

where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of n, with each n_k either 0 or 1.

We put

$$I_k^{(j)}(x) = \begin{cases} 1\ ,\ \text{if } x \in [0,1] \setminus \Delta_k^{(j)}\ , \\ 1 - 2^k\ ,\ \text{if } x \in \Delta_k^{(j)} = (\frac{j-1}{2^k}, \frac{j}{2^k})\ , \end{cases} \quad ;\ k = 1, 2, ...\ ,\ 1 \le j \le 2^k\ , \tag{2}$$

and periodically extend these functions on R^1 with period 1.

By $\chi_E(x)$ we denote the characteristic function of the set E, i.e.

$$\chi_E(x) = \begin{cases} 1\ ,\ \text{if } x \in E\ , \\ 0\ ,\ \text{if } x \notin E\ . \end{cases} \tag{3}$$

Then, clearly

$$I_k^{(j)}(x) = \varphi_0(x) - 2^k \cdot \chi_{\Delta_k^{(j)}}(x)\ , \tag{4}$$

and let for the natural numbers $k \ge 1$, and $j \in [1, 2^k]$

$$b_i(\chi_{\Delta_k^{(j)}}) = \int_0^1 \chi_{\Delta_k^{(j)}}(x)\varphi_i(x)dx = \pm\frac{1}{2^k}\ ,\ 0 \le i < 2^k \tag{5}$$

$$a_i(I_k^{(j)}) = \int_0^1 I_k^{(j)}(x)\varphi_i(x)dx = \begin{cases} 0\ ,\ \text{if } i = 0 \text{ or } i \ge 2^k\ , \\ \pm 1\ ,\ \text{if } 1 \le i < 2^k\ . \end{cases} \tag{6}$$

Hence

$$\chi_{\Delta_k^{(j)}}(x) = \sum_{i=0}^{2^k-1} b_i(\chi_{\Delta_k^{(j)}})\varphi_i(x) \tag{7}$$

$$I_k^{(j)}(x) = \sum_{i=1}^{2^k-1} a_i(I_k^{(j)})\varphi_i(x) \tag{8}$$

Lemma 1. *Let dyadic interval* $\Delta = \Delta_m^{(k)} = ((k-1)/2^m; k/2^m), \quad k \in [1, 2^m]$ *and numbers* $N_0 \in \mathbb{N},\ \gamma \neq 0,\ \epsilon \in (0,1)$ *be given. Then there exists a measurable set* $E \subset [0,1]$ *and a polynomial* Q *in the Walsh system* $\{\varphi_k\}$ *of the following form*

$$Q = \sum_{k=N_0}^{N} c_k \varphi_k$$

which satisfy the following conditions:

1) *the coefficients* $\{a_k\}_{k=N_0}^{N}$ *are* 0 *or* $\pm\gamma \mid \Delta \mid$

2) $|E| > (1-\varepsilon)|\Delta|$,

3) $Q(x) = \begin{cases} \gamma & : \quad \text{if } x \in E \\ 0 & : \quad \text{if } x \notin \Delta \end{cases}$,

4) $$\max_{N_0 \le m \le N} \left(\int_0^1 \left| \sum_{k=N_0}^{m} c_k \varphi_k(x) \right|^p dx \right)^{1/p} \le \begin{cases} A_2 |\gamma| \epsilon^{-1/2} |\Delta|^{1/2} & , \quad \text{if } p = 1 \\ A_p |\gamma| \epsilon^{-1/q} |\Delta|^{1/p} & , \quad \text{if } p > 1 \end{cases} ,$$

where A_p *is a constant depending only upon* p, *and* $\frac{1}{p} + \frac{1}{q} = 1$

Proof. Let

$$\nu_0 = \left[\log_2 \frac{1}{\epsilon}\right] + 1 \quad ; s = [\log_2 N_0] + m. \tag{9}$$

We define the polynomial $Q(x)$ and the numbers c_n, a_i and b_j in the following form:

$$Q(x) = \gamma \cdot \chi_{\Delta_m^{(k)}}(x) \cdot I_{\nu_0}^{(1)}(2^s x), \quad x \in [0;1]. \tag{10}$$

$$c_n = c_n(Q) = \int_0^1 Q(x)\varphi_n(x)dx \ , \ \forall n \ge 0, \tag{11}$$

$$b_i = b_i(\chi_{\Delta_m^{(k)}}) \ , \ 0 \le i < 2^m \ , \qquad a_j = a_j(I_{\nu_0}^{(1)}) \ , \ 0 < j < 2^{\nu_0} \ . \tag{12}$$

Taking into consideration the following equation

$$\varphi_i(x) \cdot \varphi_j(2^s x) = \varphi_{j \cdot 2^s + i}(x) \ , \text{ if } 0 \le i \ , \ j < 2^s \ \text{ (see (1))} \ ,$$

and having the following relations (5)-(8) and (10)-(12), we obtain that the polynomial $Q(x)$ has the following form:

$$Q(x) = \gamma \cdot \sum_{i=0}^{2^m - 1} b_i \varphi_i(x) \cdot \sum_{j=1}^{2^{\nu_0 - 1}} a_j \varphi_j(2^s x) =$$

$$= \gamma \cdot \sum_{j=1}^{2^{\nu_0 - 1}} a_j \cdot \sum_{i=0}^{2^m - 1} b_i \varphi_{j \cdot 2^s + i}(x) = \sum_{k=N_0}^{\bar{N}} c_k \varphi_k(x) \ , \tag{13}$$

where

$$c_k = c_k(Q) = \begin{cases} \pm\frac{\gamma}{2^m} \text{ or } 0 \ , & \text{if } k \in [N_0, \bar{N}] \\ 0 \ , & \text{if } k \notin [N_0, \bar{N}] \end{cases} \quad , \bar{N} = 2^{s+\nu_0} + 2^m - 2^s - 1 \ . \tag{14}$$

Then let

$$E = \{x; Q(x) = \gamma\} \ .$$

Clearly that (see (2) and (10)),

$$|E| = 2^{-m}(1-2^{-\nu_0}) > (1-\epsilon)|\Delta|, \tag{15}$$

$$Q(x) = \begin{cases} \gamma \text{ , if } x \in E \text{ ,} \\ \gamma(1-2^{\nu_0}) \text{ , if } x \in \Delta \setminus E \text{ ,} \\ 0 \text{ , if } x \notin \Delta \text{ .} \end{cases} \tag{16}$$

Thus, for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$\left(\int_0^1 |Q(x)|^p dx\right)^{\frac{1}{p}} \le |\gamma||\Delta|^{\frac{1}{p}} 2^{1+\frac{1}{q}} \epsilon^{-\frac{1}{q}} < 4|\gamma||\Delta|^{\frac{1}{p}} \epsilon^{-\frac{1}{q}} \text{ .}$$

Since Q is a Walsh polynomial, it is own Walsh-Fourier series; that is,

$$c_n = \int_0^1 Q(x)\varphi_n(x)dx \text{ , } \forall n \in \mathbb{N} \text{ ,}$$

thus, from Paley's theorem $||S_n(Q)|| \le \bar{A}_p ||Q||_p$, $\forall n \in \mathbb{N}$ and $p > 1$, where $\bar{A}_p$ is a constant depending only upon p. Hence

$$\max_{N_0 \le m \le N} \int_0^1 \left|\sum_{k=N_0}^{m} c_k \varphi_k(x)\right| dx \le \max_{N_0 \le m \le N} \left(\int_0^1 \left|\sum_{k=N_0}^{m} c_k \varphi_k(x)\right|^2 dx\right)^{\frac{1}{2}} \le \left(\int_0^1 Q^2(x)dx\right)^{\frac{1}{2}} \le$$

$$\le A_2|\gamma|\epsilon^{-1/2}|\Delta|^{1/2} \text{ ,}$$

and

$$\max_{N_0 \le m \le N} \left(\int_0^1 \left|\sum_{k=N_0}^{m} c_k \varphi_k(x)\right|^p dx\right)^{\frac{1}{p}} \le A_p|\gamma|\epsilon^{-1/q}|\Delta|^{1/p} \text{ , } \forall p > 1 \text{ ,}$$

where $A_p = 4\bar{A}_p$, $p > 1$.

Lemma 1 is proved.

Lemma 2. *Let given the numbers* $\tilde{N} \in \mathbb{N}$, $0 < \epsilon < 1$, $p_0 > 1$. *Then for any function* $f \in L^{p_0}(0,1)$, $\|f\|_{L_{p_0}} > 0$, *one can find a set* $E \subset [0,1]$ *and a polynomial in the Walsh system*

$$Q = \sum_{k=\tilde{N}+1}^{M} a_k \varphi_k \text{ ,}$$

satisfying the following conditions:

1) $0 \le |a_k| < \epsilon$ *and the non-zero coefficients in* $\{|a_k|\}_{k=\tilde{N}+1}^{M}$ *are in decreasing order ,*

2) $| E | > 1-\epsilon$,

3) $\left(\int_E |Q(x) - f(x)|^{p_0} dx\right)^{\frac{1}{p_0}} < \epsilon$,

4) $\max_{\tilde{N}+1 \le m \le M} \left(\int_e |\sum_{k=\tilde{N}+1}^{m} a_k \varphi_k(x)|^p dx\right)^{\frac{1}{p}} < \left(\int_e |f(x)|^p dx\right)^{\frac{1}{p}} + \epsilon$, $\forall p \le p_0$.

for every measurable subset e *of* E.

Proof. We choose some non-overlapping binary intervals $\{\Delta_\nu\}_{\nu=1}^{\nu_0}$ and a step function

$$\varphi(x)=\sum_{\nu=1}^{\nu_0}\gamma_\nu\cdot\chi_{\Delta_\nu}(x)\ ,\ \ \sum_{\nu=1}^{\nu_0}|\Delta_\nu|=1\ , \tag{17}$$

satisfying the conditions

$$\max_{1\le\nu\le\nu_0}|\gamma_\nu|\left(A_2\epsilon^{-\frac{1}{2}}|\Delta_\nu|^{\frac{1}{2}}+A_p\epsilon^{-\frac{1}{q}}|\Delta_\nu|^{\frac{1}{p}}\right)<\frac{\epsilon}{2}\ ,\ \ \forall p>1\ ,\ \frac{1}{p}+\frac{1}{q}=1\ , \tag{18}$$

$$0<|\gamma_{\nu_0}||\Delta_{\nu_0}|<...<|\gamma_\nu||\Delta_\nu|<...<|\gamma_1||\Delta_1|<\frac{\epsilon}{2}\ , \tag{19}$$

$$\left(\int_0^1|f-\varphi|^{p_0}dx\right)^{\frac{1}{p_0}}<\frac{\epsilon}{2}\ . \tag{20}$$

Successively applying Lemma 1, we determine some sets $E_\nu\subset[0,1]$ and polynomials

$$Q_\nu=\sum_{j=m_{\nu-1}}^{m_\nu-1}a_j\varphi_j\ ,\ (m_0=\bar{N}+1)\ ,\ \nu=1,...,\nu_0\ , \tag{21}$$

where $a_j=0$ or $\pm\gamma_j|\Delta_j|$, if $j\in[m_{\nu-1},m_\nu)$,

$$|E_\nu|>(1-\frac{\epsilon}{2})\cdot|\Delta_\nu|\ , \tag{22}$$

$$Q_\nu=\begin{cases}\gamma_\nu & : \ \text{if } x\in\ E_\nu\\ 0 & : \ \text{if } x\notin\ \Delta_\nu\end{cases}\ , \tag{23}$$

$$\max_{m_{\nu-1}\le m\le m_\nu-1}\left(\int_0^1\left|\sum_{k=m_{\nu-1}}^{m}a_k\varphi_k(x)\right|^p dx\right)^{\frac{1}{p}}<\begin{cases}A_2|\gamma_\nu|\varepsilon^{-1/2}|\Delta_\nu|^{\frac{1}{2}} & , \ \text{if } p=1\\ A_p|\gamma_\nu|\epsilon^{-1/q}|\Delta_\nu|^{\frac{1}{p}} & , \ \text{if } p>1\end{cases}\ . \tag{24}$$

Then let

$$E=\bigcup_{\nu=1}^{\nu_0}E_\nu\ , \tag{25}$$

$$Q=\sum_{\nu=1}^{\nu_0}Q_\nu=\sum_{k=\tilde{N}+1}^{M}a_k\varphi_k, \tag{26}$$

Let $\tilde{N}<m\le M$. From (21) and (26) we get

$$\sum_{k=\tilde{N}+1}^{m}a_k\varphi_k=\sum_{n=1}^{\nu-1}Q_n+\sum_{k=m_\nu-1}^{m}a_k\varphi_k\ \text{where } m_{\nu-1}\le m<m_\nu. \tag{27}$$

Taking into consideration that for any $x\in E$, $Q(x)=\varphi(x)$ (see (17), (23) and (26)), from (18), (24), and (27) for every measurable $e\subset E$ we obtain:

$$\left(\int_e|Q(x)-f(x)|^{p_0}dx\right)^{\frac{1}{p_0}}=\left(\int_e|\varphi(x)-f(x)|^{p_0}dx\right)^{\frac{1}{p_0}}<\epsilon.$$

$$\left(\int_e\left|\sum_{k=\tilde{N}+1}^{m}a_k\varphi_k(x)\right|^p dx\right)^{\frac{1}{p}}\le\left(\int_e\left|\sum_{n=1}^{\nu-1}\gamma_n\chi_{\Delta_n}(x)\right|^p dx\right)^{\frac{1}{p}}+\left(\int_e\left|\sum_{n=m_\nu-1}^{m}a_n\varphi_n(x)\right|^p dx\right)^{\frac{1}{p}}\le$$

$$\leq \left(\int_e |\varphi(x)|^p dx\right)^{\frac{1}{p}} + \frac{\epsilon}{2} \leq \left(\int_e |f(x)|^p dx\right)^{\frac{1}{p}} + \epsilon \ , \ \textit{for all } p \leq p_0.$$

According to (19), (21), (22) and (25) it follows

$$|E| > 1 - \epsilon$$

and $0 \leq |a_k| < \epsilon$ and the non-zero coefficients in $\{|a_k|\}_{k=\tilde{N}+1}^{M}$ are monotonically decreasing, i.e. the statements 1)- 3) of Lemma 2 are valid.

Lemma 2 is proved.

Lemma 3. *For any* $0 < \delta < 1$ *there exist a weight function* $\mu(x), 0 < \mu(x) \leq 1,$ *with* $|\ \{x \in [0,1]; \mu(x) \neq 1\}\ | < \delta$ *such that for each numbers* $p_0 > 1,\ \tilde{N} \in \mathbb{N},\ 0 < \epsilon < 1,$ *and every function* $f \in L_\mu^{p_0}(0,1),\ \|f\|_{L_{p_0}} > 0$ *, there exists a polynomial in the Walsh system of the form*

$$Q = \sum_{k=N}^{M} a_{n_k}\varphi_{n_k}\ , N > \tilde{N}$$

satisfying the following conditions:

1) $0 < |a_{n_{k+1}}| < |a_{n_k}| < \epsilon, N < k < M$,

2) $\left(\int_0^1 |Q(x) - f(x)|^{p_0}\mu(x)dx\right)^{\frac{1}{p_0}} < \epsilon$,

3) $\max\limits_{\tilde{N}+1 \leq m \leq M} \left(\int_0^1 |\sum\limits_{k=\tilde{N}+1}^{m} a_{n_k}\varphi_{n_k}(x)|^p \mu(x)dx\right)^{\frac{1}{p}} < \left(\int_0^1 |f(x)|^p\mu(x)dx\right)^{\frac{1}{p}} + \epsilon\ ,\ \forall p \leq p_0$.

Proof of Lemma 3

This is proved analogously to Lemma 3 of [6] (see pp. 9, 10).

Proof of Theorem

Let $\delta \in (0,1)$ and let

$$p_k \nearrow +\infty \ \text{ and let } \ \{f_k(x)\}_{k=1}^{\infty},\ \ x \in [0,1]\ , \tag{28}$$

be the sequence of all algebraic polynomials with rational coefficients. Applying repeatedly Lemma 3, we obtain a weight function $\mu(x)$ with $0 < \mu(x) \leq 1$ and $|\{x \in [0,1]; \mu(x) = 1\}\ | > 1 - \delta$ a sequences of polynomials in the Walsh systems $\{\varphi_n(x)\}$

$$Q_k(x) = \sum_{i=N_k}^{M_k} a_{n_i}\varphi_{n_i}(x)\ , \tag{29}$$

where

$$N_1 = 1\ ;\ N_k = M_{k-1} + 1\ ,\ k \geq 2\ ,$$

which satisfy the following conditions:

$$2^{-k} > |a_{n_i}| \geq |a_{n_{i+1}}| > 0\ ,\ \forall i \in [N_k, M_k]\ ,\ k = 1, 2, ...\ , \tag{30}$$

$$\left(\int_0^1 |Q_k(x) - f_k(x)|^{p_k}\mu(x)dx\right)^{\frac{1}{p_k}} < 2^{-4k} \ , \tag{31}$$

$$\max_{N_k \leq m \leq M_k} \left(\int_0^1 \left|\sum_{i=N_k}^{m} a_{n_i}\varphi_{n_i}(x)\right|^p \mu(x)dx\right)^{\frac{1}{p}} < \left(\int_0^1 |f_k(x)|^p\mu(x)dx\right)^{\frac{1}{p}} + 2^{-k-1} \ , \ \forall p \leq p_k \ , \tag{32}$$

Consider a series

$$\sum_{s=1}^{\infty} a_s\varphi_s(x) \ , \ \text{where } a_s = a_{n_i} \text{ if } s \in [n_i, n_{i+1}) \qquad , \tag{33}$$

Clearly (see (30), (33))

$$|a_k| \searrow 0.$$

Let $p \geq 1$ and let $f(x) \in L^p_\mu(0,1)$. We choose some $f_{\nu_1}(x)$ from sequence (28), to have

$$\left(\int_0^1 |f(x) - f_{\nu_1}(x)|^p\mu(x)dx\right)^{\frac{1}{p}} < 2^{-4} \ , \ \nu_1 > k_0 \ , \ p_{\nu_1} > p \ .$$

Suppose that the numbers $k_0 < \nu_1 < ... < \nu_{q-1}$ and polynomials $Q_{\nu_1}(x), ..., Q_{\nu_{q-1}}(x)$ are already determined satisfying to the following conditions:

$$\left(\int_0^1 |f(x) - \sum_{n=1}^{s} Q_{\nu_n}(x)|^p\mu(x)dx\right)^{\frac{1}{p}} < 2^{-4s} \ , \ s \in [2, q-1] \ , \tag{34}$$

$$\max_{N_{\nu_n} \leq m \leq M_{\nu_n}} \left(\int_0^1 \left|\sum_{i=N_{\nu_n}}^{m} a_{n_i}\varphi_{n_i}(x)\right|^p \mu(x)dx\right)^{\frac{1}{p}} < 2^{-n} \ , \ n \in [2, q-1] \tag{35}$$

Let a function $f_{\nu_q}(x)$, $\nu_q > \nu_{q-1}$ be chosen from the sequence (28) such that

$$\left(\int_0^1 \left|\left[f(x) - \sum_{j=1}^{q-1} Q_j(x)\right] - f_{\nu_q}(x)\right|^p \mu(x)dx\right)^{\frac{1}{p}} < 2^{-4(q+1)} \ . \tag{36}$$

Hence by (34) we obtain

$$\left(\int_0 |f_{\nu_q}|^p\mu(x)dx\right)^{\frac{1}{p}} < 2^{-q-1} \ . \tag{37}$$

From the conditions(31), (32), (37) follows that

$$\left(\int_0^1 |f(x) - \sum_{n=1}^{q} Q_{\nu_n}(x)|^p\mu(x)dx\right)^{\frac{1}{p}} < 2^{-4q} \ , \tag{38}$$

$$\max_{N_{\nu_q} \leq m \leq M_{\nu_q}} \left(\int_0^1 \left|\sum_{i=N_{\nu_n}}^{m} a_{n_i}\varphi_{n_i}(x)\right|^p \mu(x)dx\right)^{\frac{1}{p}} < 2^{-q} \ , \tag{39}$$

Then we obtain that the series

$$\sum_{k=1}^{\infty} \delta_k a_k \varphi_k(x) \quad \text{(see (29) and (33))}$$

where

$$\delta_k = \begin{cases} 1, \text{ if } k = n_i \text{ , where } i = \bigcup_{q=1}^{\infty} [N_{\nu_q}, M_{\nu_q}], \\ 0 \text{ , otherwise .} \end{cases}$$

converges to $f(x)$ in the$L^p_\mu(0, 1)$ - norm.

Theorem is proved.

3 Conclusion

We prove that for any $0 < \delta < 1$ there exists a measurable function $\mu(x)$, $0 < \mu(x) \leq 1$, with $| \{x \in [0,1]; \mu(x) \neq 1\} | < \delta$, and a series in the Walsh system $\{\varphi_n\}$ of the form

$$\sum_{n=0}^{\infty} a_n \varphi_n, \quad \text{with} \quad |a_n| \searrow 0,$$

such that for any $p \geq 1$ and any function $f \in L^p_\mu(0, 1)$ one can fined subseries of above series converging to f in $L^p_\mu(0, 1)$.

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Competing Interests

The authors declare that no competing interests exist.

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