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An Inferior Limit on the Number of Twin Primes up to 6 *P*

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Abstract

Because the density of non-ranks in the interval $1 ... P_j$ is smaller than the average density of non-ranks of parent primes $5 \le P_i \le P_j$ in the interval $1 ... L_j$, where $L_j = \prod_{i=3}^{j} P_i$, there is an inferior limit on the number of twin ranks up to a prime P_j, and hence on the number of twin primes up to a number $6 P_j$.

Keywords: Twin primes; twin prime conjecture; distribution of primes; sieves.

1 Introduction

In a previous paper [1] it was shown that by analyzing the twin primes from the point of view of their analogues, the twin ranks, one arrives at several characteristics potentially useful for solving the Twin Prime Conjecture [2]. The concepts of "twin ranks" and "non-ranks" were introduced in an earlier paper [3], which also specified some of their properties. Here we present an important characteristic of the distribution of the non-ranks up to a prime P, which shows that there is an inferior limit on the number of twin ranks up to P, and implies that there is an infinite number of twin primes.

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2 Definitions

- Twin prime pair: Two consecutive prime numbers P_j and P_{j+1} characterized by the equality $P_{j+1} = P_j + 2$. (Here P_j is the j^{th} prime)
- Twin index: The number $K = P_j + 1$ between a pair of twin primes P_j and P_{j+1}.
- Twin rank: A positive integer $k_* = K / 6$, where K is a twin index.
- Non-ranks: The complement of twin ranks in the set of positive integers.
- Basic number: a positive integer $M_j = \left(P_j^2 - 1\right) / 6$.
- Basic interval: A number interval between two consecutive basic numbers.
- Parent prime: The smallest prime required by eq. (1) (see below) to identify a non-rank.
- Remnants: The complement of non-ranks of parent primes $5 \le P_i \le P_j$ in an interval.
- Covering process: The operation of identifying the non-ranks in a given interval.

3 Properties

Twin indices. Since all primes P except 2 and 3 are of the form $P = \pm 1 (\text{mod}\, 6)$ all twin indices except 4 are divisible by 6.

Twin ranks. Despite their quasi-randomness, the twin ranks obey certain rules. For example:

- All numbers that end in 1, 4, 6, or 9 cannot be twin ranks;
- All twin ranks of the form $k_* = n^2$, where n is a positive integer, are divisible by 5;
- All twin ranks of the form $k_* = n^3$ are divisible by 7;

The first two of the above properties can be easily proved; the third property has not been studied yet. It will be interesting to see if there are other twin ranks of the form $k_* = \left(a\, m\right)^b$ with a determined by b and both of them different from the pairs 5, 2 and 7, 3 respectively.

Non-ranks. Unlike twin ranks whose distribution cannot be predicted, the non-ranks are distributed in an orderly manner at precise locations in the natural number series. They form an infinite number of groups and super-groups with an inner symmetry, a precise interval length and a well-defined number of terms.

- There are two non-ranks associated to a prime P symmetrically distributed at equal distances $\pm[P/6]$ from all numbers that are multiples of P. (Here [x] means "the nearest integer to x", and we are using it in this paper because the remainder from the division of any prime $P \ge 5$ by 6 is either 1/6 or 5/6 but never ½). One has:
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$$k = n\, P_i \pm [P_i / 6] \tag{1}$$

With this equation, by using all primes $5 \le P_i \le P_j$ and all necessary integers n, one can find all non-ranks smaller than a basic number $M_{j+1} = \left(P_{j+1}^2 - 1\right) / 6$. By subtracting these non-ranks from the set of positive integers $1 \ldots M_{j+1}$ one obtains all twin ranks smaller than M_{j+1}, and hence all twin primes smaller than $6M_{j+1}$.

- The covering process goes in steps from one basic interval to the next one. A number smaller than M_{j+1} uncovered after using all primes smaller than P_j (and hence shown to be a twin rank) cannot be "covered" (and hence shown to be a non-rank) by primes larger than P_j. This is because each time one goes to the next basic interval during the covering process, using a bigger prime, one starts from a larger number.
- All non-ranks of the same parent prime P_j form an infinite number of consecutive sets called "non-rank groups" of length $L_j = \prod_{i=3}^{j} P_i$ each of them containing exactly $G_j = 2\prod_{i=3}^{j-1}(P_i - 2)$ members.
- The union of all non-ranks of parent primes $5 \le P_i \le P_j$ form an infinite number of consecutive sets called "non-rank super-groups" of length L_j each of them containing exactly $S_j = L_j - \prod_{i=3}^{j}(P_i - 2)$ members. Although L_j is the length of the super-group, we will refer to it as the super-group L_j.
- All first groups and super-groups begin at $1+3=4$ and end at $L+3$. (The first three numbers $1,2,3$, which are not covered by eq. 1, are all twin ranks). As previously explained in connection with Fortune's Conjecture [4] the distribution of gaps in this type of sets repeats itself from one group or super-group to the next. Here we are interested only in the non-ranks belonging to the first super-groups.
- Because $S_j < L_j$, there always are numbers in a super-group L_j that do not belong to the set of non-ranks of parent primes $5 \le P_i \le P_j$. This set of numbers, called "remnants", have exactly $R_j = \prod_{i=3}^{j}(P_i - 2)$ members and contain twin ranks and non-ranks of parent primes larger than P_j. (We used here and above the adverb "exactly" to stress that these are not probabilistic estimates, but exact numbers).

It is very important to realize that while the number and distribution of twin ranks (and hence twin primes) are determined by non-ranks, the non-ranks do not depend in any way of the distribution or even the existence of twin primes.

4 Analysis

Let us consider the set of the non-ranks of parent primes $5 \le P_i \le P_j$ in the super-group L_j. This set contains S_j members symmetrically distributed at equal distances $\pm[P_i / 6]$ from all numbers in the interval that are multiple of one of the primes $5 \le P_i \le P_j$. Because of this orderly distribution, their density in a large enough subset does not differ too much from the average density

$$\sigma_j = \frac{S_j}{L_j} = 1 - \prod_{i=3}^{j} \frac{P_i - 2}{P_i} \tag{2}$$

One can approximate, therefore, the number of non-ranks in an interval inside L_j by the product

$$N_{kI} = \sigma_j I \tag{3}$$

Of course, this is only an approximation; the actual value can be smaller or it can be larger, we don't know. However, if one wants to approximate the number of non-ranks in the interval $1...P_j$ in the same manner and writes

$$N_{kp} = \sigma_j P_j \tag{4}$$

one always obtains a number larger than the actual value. This is because eq. (4) takes into account all primes $5 \le P_i \le P_j$, despite the fact that one needs only the first primes up to $P_a \approx \sqrt{6P_j + 1} << P_j$ to cover the interval $1...P_j$. All non-ranks of parent primes larger than P_a fall outside of the interval $1...P_j$. Example:

Let $j = 9$ and $I = 200$. One has $P_j = 23$, $L_j = \prod_{i=3}^{9} P_i = 37182145$, $S_j = L_j - \prod_{i=3}^{9}(P_i - 2) = 29229970$ and $\sigma_j = S_j / L_j \cong 0.78613$. The approximate number of non-ranks of parent primes $5 \le P_i \le P_j$ in the interval $1...200$ is $N_{kI} = 157 \approx \sigma_j I$. The actual number is 158. The approximate number of non-ranks in the interval $1...23$ is $N_{kp} = 18 \approx \sigma_j P_j$. The actual number is 13. They are: 4, 6, 8, 9, 11, 13, 14, 15, 16, 19, 20, 21, 22. None of these non-ranks is of a parent prime larger than $P_a = 13 \approx \sqrt{6P_j + 1}$.

As mentioned, by subtracting the non-ranks of parent primes $5 \le P_i \le P_j$ from the set of positive integers $1...M_{j+1}$, where $M_{j+1} = \left(P_{j+1}^2 - 1\right)/6$, one obtains all twin ranks smaller than M_{j+1}, and hence all twin primes smaller than $6M_{j+1}$. But the interval $1...P_j$ is well inside the interval $1...M_{j+1}$. Therefore, one can approximate the number of twin ranks up to P_j by $T_{rp} = P_j - N_{kp}$. Since N_{kp} is always larger than the actual value, this approximation is always smaller than the real value. With the help of eqs. (2) and (4), one obtains

$$T_{rp} = P_j \prod_{i=3}^{j} \frac{P_i - 2}{P_i} \tag{5}$$

With $R_j = \prod_{i=3}^{j}(P_i - 2)$, it is easy to see that the product on the right hand side represents the average density of remnants of parent primes $5 \le P_i \le P_j$ in the super-group L_j. Based on all these considerations one can write the following.

Theorem: Given a prime P_j, the number of twin ranks up to P_j is larger than P_j multiplied by the average density of remnants of parent primes $5 \le P_i \le P_j$ in the corresponding super-group.
Since there is a one-to-one correspondence between the twin ranks in an interval $a...b$ and the twin indices in the interval $6a...6b$, it follows that:

Corollary: Given a prime P_j, the number of twin prime pairs up to $6P_j$ is larger than P_j multiplied by the average density of remnants of parent primes $5 \le P_i \le P_j$ in the corresponding super-group.

From Mertens' 3rd theorem [5]

$$\prod_{i=1}^{j} \frac{P_i - 1}{P_i} \approx \frac{e^{-\gamma}}{\ln P_j} \tag{6}$$

Where $\gamma \approx 0.57721$ is the Euler 's constant [6], and the twin prime constant [7]

$$C_2 = \prod_{i=2}^{\infty} \frac{P_i(P_i - 2)}{(P_i - 1)^2} \approx 0.66016... \tag{7}$$

one obtains

$$T_{rp} \approx 12\, C_2\, P_j \left(\frac{e^{-\gamma}}{\ln P_j} \right)^2 \tag{8}$$

The factor of 12 in the above equation comes from the different values of the starting prime index in the products in eqs. (5), (6), and (7), vis. $i = 3,\ 1,\ 2$, respectively.

According to the so called Hardy-Littlewood" conjecture B" [7], the number of twin primes pairs up to a number $6P_j$ (and, hence, of twin ranks up to P_j) is on the order of

$$\pi_2(6P_j) \approx 2C_2 \frac{6P_j}{\ln^2 n} \tag{9}$$

Interestingly, despite the fact that we used a completely different approach and did not resort to any of the considerations that lead Hardy and Littlewood to this conjecture, for large primes the ratio of the two approximations is constant and on the order of $e^{-2\gamma}$.

5 Results

j = {10, 100, 1000, 10000, 100000, 1000000, 10000000};

P_j = {29, 541, 7919, 104729, 1299709, 15485863, 179424673};

T_{rp} = {6, 34, 245, 1956, 16375, 141096, 1240488};

$\pi_2\left(6P_j\right)$ = {11, 83, 668, 5539, 47540, 412592, 3659077};

Clearly there are more twin ranks up to a certain prime P than the inferior limit T_{rp} .

6 Concluding Remarks

Perhaps the most important characteristic of this paper regarding the twin primes is the fact that it did not resort to any probabilistic estimates. All relevant quantities are either exact integers or ratios of two such integers. This fact was possible because we shifted the focus from the disordered distribution of twin primes to the organized distribution of non-ranks. In this way we were able to link the density of twin ranks in an interval (a characteristic for which we did not have a reliable approximation) with the density of non-ranks in another interval, for which we had a precise formula.

Since the limit T_{rp} tends to infinity as one goes up on the number series, the inescapable conclusion is that there are infinitely many twin primes.

Competing Interests

Author has declared that no competing interests exist.

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