British Journal of Mathematics & Computer Science
7(2): 150-167, 2015, Article no.BJMCS.2015.111
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Almost Convergent Sequence Space Derived by Generalized Fibonacci Matrix and Fibonacci Core

Murat Candan[1*] **and Kuddusi Kayaduman**[2]

[1] *Faculty of Arts Sciences, Department of Mathematics, İnönü University, Malatya-44280, Turkey.*
[2] *Faculty of Arts Sciences, Department of Mathematics, Gaziantep University, Gaziantep-27310, Turkey.*

Article Information
DOI: 10.9734/BJMCS/2015/15923
Editor(s):
(1) Radko Mesiar, Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology Bratislava, Slovakia.
Reviewers:
(1) W. Obeng-Denteh, Mathematics Department, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana.
(2) Anonymous, Turkey.
Complete Peer review History:
http://www.sciencedomain.org/review-history.php?iid=933&id=6&aid=8084

Original Research Article

Received: 26 December 2014
Accepted: 19 January 2015
Published: 10 February 2015

Abstract

Considerable interest in this article is to introduce the sequence space $\widehat{c}^{f(r,s)}$ derived by generalized difference Fibonacci matrix in which $r, s \in \mathbb{R} \setminus \{0\}$, also to discuss and compare with some well-known spaces defined previously. In addition to those, after demonstrating that the spaces $\widehat{c}^{f(r,s)}$ and $\widehat{c}$ are linearly isomorphic, we have determined the $\beta-$ and $\gamma-$duals of space $\widehat{c}^{f(r,s)}$ and have characterized some matrix classes on this space. As a conclusion, we have also found out that the space has not a Schauder basis. Lastly, we have presented the Fibonacci core of a complex-valued sequence and deal with inclusion theorems with respect to Fibonacci core type.

Keywords: Sequence spaces, almost convergence, Fibonacci matrix, β-dual, matrix transformations, core theorems.

2010 Mathematics Subject Classification: 46A45; 40A05; 46A35.

1 Introduction

By w, we denote the space of all real or complex-valued sequences $x = (x_k)$. Any vector subspace of w is called a sequence space. As usual, we write $c_0,\ c$ and l_∞ denote the sets of sequences that

**Corresponding author: E-mail: murat.candan@inonu.edu.tr*

are convergent to zero, convergent and bounded, respectively. In addition to these, the symbols bs and cs are known the spaces of all bounded and convergent series, respectively.

The almost convergence has fundamental importance for this article. So, the concept is stated in this paragraph. The class $\widehat{c}$ of almost convergent sequences was introduced by G.G. Lorentz [1], using the idea of the Banach limits. A Banach limit L is defined on ℓ_∞, as a non–negative linear functional, such that $L(\varphi x) = L(x)$ and $L(e) = 1$, where φ is shift operator and $e = (1, 1, \cdots, 1 \cdots)$. The existence of Banach limits was proven by Banach [2] in his book. A sequence $x = (x_k) \in \ell_\infty$ is known to be almost convergent to the generalized limit α if all Banach limits of x is α [1], and denoted by $\widehat{c} - \lim\, x_k = \alpha$. Let φ^j be the composition of φ with itself j times and define $t_{mn}(x)$ for a sequence $x = (x_k)$ by

$$t_{mn}(x) := \frac{1}{m+1} \sum_{j=0}^{m} \varphi_n^j(x) \text{ for all } m, n \in \mathbb{N}.$$

Lorentz [1] proved that $\widehat{c} - \lim\, x_k = \alpha$ iff $\lim_{m\to\infty} t_{mn}(x) = \alpha$, uniformly in n. It is well–known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. As mentioned in the above, by $\widehat{c}_0$ and $\widehat{c}$, we denote the space of all almost null and almost convergent sequences, that is

$$\begin{aligned} \widehat{c}_0 &:= \left\{ x = (x_k) \in \ell_\infty : \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n \right\}, \\ \widehat{c} &:= \left\{ x = (x_k) \in \ell_\infty : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \alpha \text{ uniformly in } n \right\}. \end{aligned}$$

It is known that $\hat{c}$ is a Banach space with the norm [3]

$$\|x\|_{\widehat{c}} = \sup_{m,n\in\mathbb{N}} \left| \sum_{j=0}^{m} \frac{x_{n+k}}{m+1} \right|.$$

Another notion we need is that of matrix transformation. For this reason, in this paragraph, we shall be concerned with matrix transformation from a sequence space X to a sequence space Y. Given any infinite matrix $A = (a_{nk})$ of real numbers a_{nk}, where $n,\ k \in \mathbb{N}$, any sequence x, we write $Ax = \big((Ax)_n\big)$, the A-transform of x, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. If $x \in X$ implies that $Ax \in Y$ then we say that A defines a matrix mapping from X into Y and denote it by $A : X \to Y$. By $(X : Y)$, we mean the class of all infinite matrices such that $A : X \to Y$.

When X and Y have limits $X - \lim$ and $Y - \lim$, respectively, and for all $x \in X$, $A \in (X : Y)$ and $Y - \lim_n A_n(x) = X - \lim_k x_k$ is valid; we have the right to say that A regularly maps X into Y and also shown it as $A \in (X : Y)_{reg}$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x. Further, a triangle U uniquely has an inverse $U^{-1} = V$ that is also a triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$.

For an arbitrary sequence space μ, μ_A is known the domain an infinite matrix A as

$$\mu_A = \{x = x_k \in \omega : Ax \in \mu\}.$$

Since μ_A is a linear subspace of the space w of all real or complex-valued sequences, it is also a sequence space. In recent years, the approach to construct a new sequence space by means of the matrix domain of a particular triangle has been used by some of the writers in many research articles [4, 5, 6, 7, 8]. For an overview of the literature on new almost convergent sequence space, see [9, 10, 11, 12] and the references therein. Since we are motivated by the references, especially the spaces $\widehat{c}_{R^t}$, $\widehat{c}_C$, $\widehat{c}_{B(r,s)}$, and $\widehat{c}_{B(\widetilde{r},\widetilde{s})}$ have been studied in [9, 10, 11, 12], respectively, where R^t is the Riesz mean, C is the Cesàro matrix of order one, $B(r,s) = \{b_{nk}(r,s)\}$ and $B(\widetilde{r},\widetilde{s}) = \{b_{nk}(r_n,s_n)\}$ are the generalized difference matrix and double sequential band matrix, respectively defined by

$$b_{nk}(r,s) = \begin{cases} r & , \quad k=n, \\ s & , \quad k=n-1, \\ 0 & , \quad \text{otherwise} \end{cases} \quad b_{nk}(\widetilde{r},\widetilde{s}) = \begin{cases} r_n & , \quad k=n, \\ s_n & , \quad k=n-1, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k,n \in \mathbb{N}$, where $r,s \in \mathbb{R}\backslash\{0\}$ and $\tilde{r} = (r_n)_{n=0}^{\infty}$ and $\tilde{s} = (s_n)_{n=0}^{\infty}$ be given convergent sequences of positive real numbers.

In recent years, Kara and Elmaağaç [13] defined and examined $u-$difference almost sequence space $\hat{c}^u = (\hat{c})_{A^u}$, where $A^u = (a^u_{nk})$ denote $u-$difference matrix. To write in a more clear way,

$$a^u_{nk} = \begin{cases} (-1)^{n-k}u_k & , \quad n-1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $k,n \in \mathbb{N}$. Purely for the development of almost convergence and some generalizations, the excellent results [14, 15, 16, 17, 18, 19, 20, 21, 22, 23], are recommended.

The plan of the present paper is organized as follows. After collecting all the necessary definitions and results, we have firstly introduced new sequence space $\widehat{c}^{f(r,s)}$ under the domain of the matrix $\widehat{F}(r,s)$, constituted by using Fibonacci sequences and non-zero real numbers r and s, of $\widehat{c}$ previously defined. Later, we give some inclusion theorems and demonstrate that $\widehat{c}^{f(r,s)}$ is linearly isomorphic to the space $\widehat{c}$. As a conclusion, we also show that the newly defined space has not a Schauder basis and determine the $\beta-$ and $\gamma-$duals of the space $\widehat{c}^{f(r,s)}$ and characterize the classes of infinite matrices related to sequence space $\widehat{c}^{f(r,s)}$. In the last section, we have defined $B_{\widehat{F}(r,s)} - core$ of a sequence and characterized certain class of matrices for which $B_{\widehat{F}(r,s)} - core(Ax) \subseteq \mathcal{K} - core(x)$, $\mathcal{K} - core(Ax) \subseteq B_{\widehat{F}(r,s)} - core(x)$, $B_{\widehat{F}(r,s)} - core(Ax) \subseteq B_{\widehat{F}(r,s)} - core(x)$ and $B_{\widehat{F}(r,s)} - core(Ax) \subseteq st - core(x)$ for all $x \in \ell_\infty$.

2 The Sequence Space $\widehat{c}^{f(r,s)}$ Derived by the Domain of the Matrix $\widehat{F}(r,s)$

In this subsection, before stating the new almost sequence space derived generalized difference matrix which established both Fibonacci sequences and $r,s \in \mathbb{R}$, we present some historical information about Fibonacci sequences. In 1202, the Fibonacci numbers first came out in the book *"Liber Abaci"*, which means *"The Book of Calculation"* among the first western books was a historic book on arithmetic written by Leonardo of Pisa, commonly known as Fibonacci. There are many ways to introduce the Fibonacci sequence, each of which is an equivalent way of defining the same thing. Here, let us explain this concept. A numeric sequence is a set of ordered numbers generated by well-defined algorithm. The easiest method of generating a number sequence is to use one or two kernel values and an suitable recursive equation. One of the most well-known number sequence is Fibonacci sequence. This sequence is obtained by the following recursive formula

$$f_n = f_{n-1} + f_{n-2} \text{ with } n \geq 2.$$

That is, each term in the sequence is equal to the sum of the previous two terms. This sequence requires the kernel values f_0 and f_1. Throughout our study, we will take f_0 and f_1 as 1.

Now, we are taking a look at some of the famous properties such as *Golden Ratio*, and *Cassini formula* of the Fibonacci sequence [24].

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n}=\frac{1+\sqrt{5}}{2}=\varphi \text{ (Golden Ratio)},$$
$$\sum_{k=0}^{n} f_k = f_{n+2}-1 \text{ for each } n\in\mathbb{N},$$
$$\sum_k \frac{1}{f_k} \text{ converges},$$
$$f_{n-1}f_{n+1}-f_n^2=(-1)^{n+1} \text{ for all } n\geq 1 \text{ (Cassini Formula)}.$$

It can easily be derived by replacing f_{n+1} in Cassini's formula namely $f_{n-1}^2+f_nf_{n-1}-f_n^2=(-1)^{n+1}$.

Many authors used the Fibonacci numbers to establish a sequence space. In particular, we would like to mentioned certain results. Kara [25] defined the sequence space $\ell_p(\widehat{F})$ as follows:

$$\ell_p(\widehat{F})=\left\{x\in\omega : \widehat{F}x\in\ell_p\right\},\ \ (1\leq p\leq\infty),$$

where $\widehat{F}=(\widehat{f}_{nk})$ is the double band matrix defined by the sequence (f_n) of Fibonacci numbers as follows

$$\widehat{f}_{nk}=\begin{cases} -\frac{f_{n+1}}{f_n} & , \ k=n-1, \\ \frac{f_n}{f_{n+1}} & , \ k=n, \\ 0 & , \ 0\leq k<n-1 \text{ or } k>n \end{cases}$$

for all $k,n\in\mathbb{N}$. Also, Kara et al. [26] characterized some classes of compact operators on the spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$, where $1\leq p<\infty$. Furthermore, the sequence spaces $\lambda(\widehat{F})$ and $\mu(\widehat{F},p)$ are studied by Başarır et al. [27], and Kara and Demiriz [28], respectively, where $\lambda\in\{c_0,c\}$ and $\mu\in\{c_0,c,\ell_\infty\}$. Recently, Candan [29] has introduced the sequence spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ after then, Candan and Kara [30] have examined the space $\ell_p(\widehat{F}(r,s))$ in which $1\leq p\leq\infty$ and the matrix $\widehat{F}(r,s)=(\widehat{f}_{nk}(r,s))$ constituted by using Fibonacci sequences and non-zero real numbers r and s, i.e.,

$$\widehat{f}_{nk}(r,s)=\begin{cases} s\frac{f_{n+1}}{f_n} & , \ k=n-1, \\ r\frac{f_n}{f_{n+1}} & , \ k=n, \\ 0 & , \ 0\leq k<n-1 \text{ or } k>n. \end{cases}$$

Now, we define the sequence space $\widehat{c}^{f(r,s)}$ and give an isomorphism between the spaces $\widehat{c}^{f(r,s)}$ and $\widehat{c}$ respectively. Later, we determine the $\beta-$dual of the space $\widehat{c}^{f(r,s)}$.

We introduce the sequence space $\widehat{c}^{f(r,s)}$ as the set of all sequence whose $\widehat{F}(r,s)$ transforms are in the space $\widehat{c}$, that is

$$\widehat{c}^{f(r,s)}=\left\{x=x_k\in\ell_\infty : \exists\alpha\in\mathbb{C}\ni \lim_{m\to\infty}\sum_{j=0}^{m}\frac{y_{n+j}}{m+1}=\alpha \text{ uniformly in } n\right\},$$

where $y=(y_n)$ is the $\widehat{F}(r,s)$-transform of a sequence $x=(x_n)$, i.e.,

$$y_n=\widehat{F}(r,s)_n(x)=\begin{cases} rx_0 & , \ n=0, \\ r\frac{f_n}{f_{n+1}}x_n+s\frac{f_{n+1}}{f_n}x_{n-1} & , \ n\geq 1 \end{cases}. \tag{2.1}$$

It is clear that the space $\widehat{c}^{f(r,s)}$ can be redefined as

$$\widehat{c}^{f(r,s)} = \widehat{c}_{\widehat{F}(r,s)}.$$

When $\alpha = 0$, we will particularly denote the space $\widehat{c}^{f(r,s)}$ by symbol $\widehat{c}_0^{f(r,s)}$.

We should state here that the matrix $\widehat{F}(r,s)$ can be reduced to the matrix $\widehat{F}$ in case $r = 1$ and $s = -1$. Therefore, the results related to the space $\widehat{c}^{f(r,s)}$ are more general and more comprehensive than the corresponding consequences of the space $\widehat{c}^{f}$ more recently defined by Demiriz et al. in [31].

For our later use we recall the following two lemmas here.

Lemma 2.1. *[32] An infinite matrix $A = (a_{nk})$ transforms each almost convergent sequence into an almost convergent sequence if and only if*

$$\|A\| = \sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty$$

$$\widehat{c} - \lim_{n\to\infty} \sum_k a_{nk} = \alpha$$

$$\widehat{c} - \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}$$

$$\lim_{q\to\infty} \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} (a_{n+i,k-1} - \alpha_{k-1} + \alpha_k - a_{n+i,k}) \right| = 0 \text{ uniformly in } n.$$

Lemma 2.2. *[33] $A = (a_{nk}) \in (\widehat{c} : \ell_\infty)$ if and only if $\|A\| < \infty$ holds.*

Theorem 2.3. *The sequence spaces $\widehat{c}_0^{f(r,s)}$ and $\widehat{c}^{f(r,s)}$ strictly include the spaces $\widehat{c}_0$ and $\widehat{c}$, respectively.*

Proof. Since the matrix $\widehat{F}(r,s)$ satisfies the conditions of Lemma 2.1, it belongs to the class $(\widehat{c} : \widehat{c})$. So, $\widehat{F}(r,s)x \in \widehat{c}_0$ or $\widehat{F}(r,s)x \in \widehat{c}$ whenever $x \in \widehat{c}_0$ or $x \in \widehat{c}$ which shows that the inclusions $\widehat{c}_0 \subset \widehat{c}_0^{f(r,s)}$ and $\widehat{c} \subset \widehat{c}^{f(r,s)}$ hold.

Let us define the sequence $(k_n) = \left(\frac{(-s)^n f^2(n+1)}{r^{n+1}}\right)$ for all $n \in \mathbb{N}$. If $\lambda \in \{\widehat{c}_0, \widehat{c}\}$, then $\widehat{F}(r,s)(k_n) = (1, 0, \cdots, 0, \cdots) = e^0 \in \lambda$. Hence, $(k_n) \in \lambda_{\widehat{F}(r,s)} \setminus \lambda$. This means that the inclusions $\widehat{c}_0 \subset \widehat{c}_0^{f(r,s)}$ and $\widehat{c} \subset \widehat{c}^{f(r,s)}$ are strict. ☐

Theorem 2.4. *If $|s/r| < 1/4$ then the inclusions $\widehat{c}_0^{f(r,s)} \subset \ell_\infty$ and $\widehat{c}^{f(r,s)} \subset \ell_\infty$ strictly hold.*

Proof. To verify the validity of the inclusion $\widehat{c}^{f(r,s)} \subset \ell_\infty$, let us assume that $|s/r| < 1/4$ and take an arbitrary $x \in \widehat{c}^{f(r,s)}$. Then, $y = \widehat{F}(r,s)x \in \widehat{c} \subset \ell_\infty$. The inverse matrix $\widehat{F}^{-1}(r,s)$ satisfies the condition of Lemma 2.2, indeed

$$\|A\| = \sup_{n\in\mathbb{N}} \sum_k \left| f_{nk}^{-1}(r,s) \right| \leq \frac{1}{inf_{n\in\mathbb{N}} \frac{rf_n}{f_{n+1}}} \sum_k \left(\frac{sup_{n\in\mathbb{N}} \frac{sf_{n+2}}{f_{n+1}}}{inf_{n\in\mathbb{N}} \frac{rf_n}{f_{n+1}}} \right)^k \leq \frac{2}{r} \sum_k \left(\frac{4s}{r} \right)^k < \infty,$$

it belongs to the class $(\widehat{c} : \ell_\infty)$ by virtue of assumption. So, $x = F^{-1}(r,s)y \in \ell_\infty$. Hence, the inclusion $\widehat{c}^{f(r,s)} \subset \ell_\infty$ holds.

Let $|s/r| \geq 1/4$. Let us consider the bounded sequence $u = (u_k)$ defined by $u = (0, \ldots, 0, 1, \ldots, 1,$

$0,\ldots,0,1,\ldots,1,\ldots)$, where the blocks 0's are increasing by factors of 100 and the blocks of 1's are increasing by factors of 10 (cf. Miller and Orhan [34]). Then, the sequence $\widehat{F}(r,s)u$ is not almost convergent. This shows that $u \in \ell_\infty \setminus \widehat{c}^{f(r,s)}$ which means that the inclusion $\widehat{c}^{f(r,s)} \subset \ell_\infty$ strictly holds.

One can show by analogy that the inclusion $\widehat{c}_0^{f(r,s)} \subset \ell_\infty$ strictly holds. So, we omit the detail. $\square$

Now, we may give following theorem concerning the isomorphism between the spaces $\widehat{c}^{f(r,s)}$ and $\widehat{c}$.

Theorem 2.5. *The sequence space $\widehat{c}^{f(r,s)}$ is linearly isomorphic to the space $\widehat{c}$, that is, $\widehat{c}^{f(r,s)} \cong \widehat{c}$.*

Proof. Before we embark in proving the theorem, we need to be sure the transformation L exists between the spaces $\widehat{c}^{f(r,s)}$ and $\widehat{c}$. For this purpose, let us take the transformation L mentioned above, with the help of the notation of (2.1) from the space $\widehat{c}^{f(r,s)}$ to the space $\widehat{c}$ by $x \longmapsto y = Lx = \widehat{F}(r,s)x$. Since it is clear to show that both L is linear and injective, we omit the details.

To prove that the transformation L is surjective, we firstly consider an arbitrary sequence $y = (y_k) \in \widehat{c}$ and later obtain the following sequence $x = (x_k)$ using the inverse matrix $\widehat{F}^{-1}(r,s)$ as follows

$$x_k = \{\widehat{F}^{-1}(r,s)y\}_k = \sum_{j=0}^{k} \frac{1}{r}\left(\frac{-s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \tag{2.2}$$

for all $k \in \mathbb{N}$. When we use the sequence (x_k) derived only just come out, we easily get

$$\begin{aligned} r\frac{f_k}{f_{k+1}}x_k + s\frac{f_{k+1}}{f_k}x_{k-1} &= r\frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{1}{r}\left(\frac{-s}{r}\right)^{k-j} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \\ &\quad + s\frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{1}{r}\left(\frac{-s}{r}\right)^{k-j-1} \frac{f_k^2}{f_j f_{j+1}} y_j = y_k \end{aligned}$$

for all $k \in \mathbb{N}$ which results in the fact that

$$\begin{aligned} \lim_{m\to\infty} \sum_{j=0}^{m} \frac{r\frac{f_{k+j}}{f_{k+1+j}}x_{k+j} + s\frac{f_{k+1+j}}{f_{k+j}}x_{k-1+j}}{m+1} &= \lim_{m\to\infty} \frac{1}{m+1}\sum_{j=0}^{m} y_{k+j} \text{ uniformly in } k \\ &= \hat{c} - \lim y_k. \end{aligned}$$

This briefly tells us that $x = (x_k) \in \widehat{c}^{f(r,s)}$. Namely L is surjective. As a conclusion, L is a linear bijection, which means that the spaces $\widehat{c}^{f(r,s)}$ and $\widehat{c}$ are linearly isomorphic. This marks the end of the proof. $\square$

We now collect some elementary important facts related to Schauder bases which will be used in the proof of the next corollary.

Remark 2.6. *[35, Remark 2.4] The matrix domain μ_A of a linear metric sequence space μ has a basis iff μ has a basis.*

Lemma 2.7. *[11, Corollarry 3.3] The Banach space $\widehat{c}$ has no Schauder basis.*

Corollary 2.8. *The space $\widehat{c}^{f(r,s)}$ has no Schauder basis.*

Proof. The proof can easily be obtained from Remark 2.6 using the fact that not only the matrix $\widehat{F}(r,s)$ is a triangle but also the space $\widehat{c}$ has not got a Schauder basis in view of Lemma 2.7. $\square$

In this paragraph, let us firstly define $S(\lambda,\mu)$ multiplier space of any sequence spaces λ and μ. If $\lambda,\mu\subset w$ and z arbitrary sequence, we can write

$$z^{-1}*\lambda=\{x=(x_k)\in w : xz\in\lambda\}$$

and

$$S(\lambda,\mu)=\cap_{x\in\lambda}x^{-1}*\mu.$$

We then go on define $\alpha-,\beta-$ and $\gamma-$ duals of an arbitrary space λ. If we choose $\mu=\ell_1$, cs and bs, then we obtain the $\alpha-,\beta-$ and $\gamma-$ duals of the space λ, respectively as

$$\lambda^{\alpha}=S(\lambda,\ell_1)=\{a=(a_k)\in w : ax=(a_kx_k)\in\ell_1 \text{ for all } x\in\lambda\}$$

$$\lambda^{\beta}=S(\lambda,cs)=\{a=(a_k)\in w : ax=(a_kx_k)\in cs \text{ for all } x\in\lambda\}$$

$$\lambda^{\gamma}=S(\lambda,bs)=\{a=(a_k)\in w : ax=(a_kx_k)\in bs \text{ for all } x\in\lambda\}.$$

The following lemma is essential to compute $\beta-$dual of the space $\widehat{c}^{f(r,s)}$.

Lemma 2.9. *[36] $A=(a_{nk})\in(\widehat{c}:c)$ if and only if there are $\alpha_k,\alpha\in\mathbb{C}$ such that*

$$\lim_{n\to\infty}a_{nk}=\alpha_k \text{ for each } k\in\mathbb{N}, \tag{2.3}$$

$$\lim_{n\to\infty}\sum_k a_{nk}=\alpha, \tag{2.4}$$

$$\lim_{n\to\infty}\sum_k\left|\Delta\left(a_{nk}-\alpha_k\right)\right|=0, \tag{2.5}$$

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|<\infty, \tag{2.6}$$

where $\Delta\left(a_{nk}-\alpha_k\right)=(a_{nk}-\alpha_k)-(a_{n,k+1}-\alpha_{k+1})$ $(n,k\in\mathbb{N})$.

Theorem 2.10. *Define the sets $d_1(r,s)$, $d_2(r,s)$, $d_3(r,s)$, $d_4(r,s)$ and $d_5(r,s)$ by*

$$d_1(r,s)=\left\{a=(\alpha_k)\in\omega : \lim_{n\to\infty}\sum_{j=k}^{n}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_j \text{ exists}\right\},$$

$$d_2(r,s)=\left\{a=(\alpha_k)\in\omega : \lim_{n\to\infty}\sum_{k=0}^{n}\left(\sum_{j=k}^{n}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_j\right) \text{ exists}\right\},$$

$$d_3(r,s)=\left\{a=(\alpha_k)\in\omega : \lim_{n\to\infty}\sum_{k=0}^{n}\left|\sum_{i=k+1}^{\infty}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{i+1}^2}{f_kf_{k+1}}a_i\right|=0\right\},$$

$$d_4(r,s)=\left\{a=(a_k)\in\omega : \lim_{n\to\infty}\sum_{k=n+1}^{\infty}|\varrho(r,s,f_k,f_{k+1},f_{k+2},a_k)|=0\right\},$$

where

$$\varrho(r,s,f_k,f_{k+1},f_{k+2},a_k) = r\frac{f_k}{f_{k+1}}a_k + \left(1+\frac{rf_k}{sf_{k+2}}\right)\sum_{i=k+1}^{\infty}\frac{1}{r}\left(\frac{-s}{r}\right)^{k-j}\frac{f_{i+1}^2}{f_kf_{k+1}}a_i$$

and

$$d_5(r,s) = \left\{a=(\alpha_k)\in\omega : \sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_j\right|<\infty\right\}.$$

Then,

$$\left\{\widehat{c}^f\right\}^{\beta} = \cap_{i=1}^5 d_i(r,s).$$

Proof. Although the technical details are somewhat involved, the idea of the proof is quite simple, however, the treatment of the details given here. Consider an arbitrary sequence $a=(a_k)\in\omega$. In that case, we get the following equalities with the help of (2.2)

$$\begin{aligned}\sum_{k=0}^{n}a_kx_k &= \sum_{k=0}^{n}a_k\left(\sum_{j=0}^{k}\frac{1}{r}\left(\frac{-s}{r}\right)^{k-j}\frac{f_{k+1}^2}{f_jf_{j+1}}y_j\right)\\ &= \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_j\right)y_k \\ &= E_n(y),\end{aligned} \tag{2.7}$$

for all $n\in\mathbb{N}$, where $E=(e_{nk})$ is defined by

$$e_{nk} = \begin{cases}\sum_{j=k}^{n}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_j & (0\le k\le n)\\ 0 & (k>n)\end{cases};\quad n,k\in\mathbb{N}.$$

Then, it is easily observed from the approach we followed above i.e., from (2.7) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in\widehat{c}^{f(r,s)}$ iff $Ey\in c$ whenever $y=(y_k)\in\widehat{c}$. Thus, we obtain from Lemma 2.9 that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in\widehat{c}^{f(r,s)}$ iff $a=(a_k)\in\cap_{i=1}^5 d_i(r,s)$. This gives that $\left\{c^f\right\}^{\beta}=\cap_{i=1}^5 d_i(r,s)$. In fact, this is exactly what we want to prove. ☐

Theorem 2.11. *The $\gamma-$dual of the space $\widehat{c}^{f(r,s)}$ is the set $d_5(r,s)$.*

Proof. The basic idea of the proof is the same as in the way of Theorem 2.10. The only difference is put the space of all bounded series bs instead of the space of all convergent series cs. ☐

3 Some Matrix Transformations Related to the Sequence Space $\widehat{c}^{f(r,s)}$

In this section, the study will be focused on the characterize the matrix transformations from $\widehat{c}^{f(r,s)}$ into any given sequence space X and from a given sequence space X into $\widehat{c}^{f(r,s)}$.

For the sake of simplicity, here and in what follows, we will write that

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{j+1}^2}{f_kf_{k+1}}a_{nj},$$

$$\bar{a}_{nk} = r\frac{f_n}{f_{n+1}}a_{nk} + s\frac{f_{n+1}}{f_n}a_{n-1,k},$$

$$a(n,k) = \sum_{j=0}^{n} a_{jk}$$

and

$$a(n,k,m) = \frac{1}{m+1}\sum_{j=0}^{m} a_{n+j,k} \quad (m \in \mathbb{N})$$

for all $k, n \in \mathbb{N}$. As $\widehat{c}^{f(r,s)} \cong \widehat{c}$, it is obvious that the equivalence "$x \in \widehat{c}^{f(r,s)}$ if and only if $y \in \widehat{c}$" holds.

Now, let us state the following two theorems to determine matrix classes on the space $\widehat{c}^{f(r,s)}$.

Theorem 3.1. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $T = (t_{nk})$ are connected with the relation*

$$t_{nk} = \tilde{a}_{nk} \tag{3.1}$$

for all $k, n \in \mathbb{N}$ and X be any given sequence space. Then, $A \in \left(\widehat{c}^{f(r,s)} : X\right)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{c}^{f(r,s)}\right\}^{\beta}$ for all $n \in \mathbb{N}$ and $T \in (\widehat{c} : X)$.

Proof. First of all, we know that the spaces $\widehat{c}^{f(r,s)}$ and $\widehat{c}$ are linearly isomorphic from Theorem 2.5. In order to prove the theorem, we will follow the same analysis employed before Başar and Kirişçi [11]. To do this, let us suppose that both X be a sequence space and condition (3.1) is valid for the matrices $A = (a_{nk})$ and $T = (t_{nk})$.

In proving necessity, we assume that $A \in \left(\widehat{c}^{f(r,s)} : X\right)$ and take any sequence $y = (y_k) \in \widehat{c}$. Under these assumptions, it is clear that $T\widehat{F}(r,s)$ exist and $\{a_{nk}\}_{k\in\mathbb{N}} \in \cap_{i=1}^{5} d_i$ So, $\{t_{nk}\}_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. In that case, Ty exist and we easily get

$$\sum_k t_{nk}y_k = \sum_k a_{nk}x_k \text{ for all } n \in \mathbb{N}$$

when on account of condition (3.1) . Newly obtained formula says us $Ty = Ax$, which clearly indicates that $T \in (\widehat{c} : Y)$.

The arguments we use in proving sufficiency are $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{c}^{f(r,s)}\right\}^{\beta}$ for all $n \in \mathbb{N}$ and $T \in (\widehat{c} : X)$ and a taken sequence $x = (x_k) \in \widehat{c}^{f(r,s)}$. By our assumption, clearly Ax exists. Using a simple calculus, we can derive the following equality

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m}\left[\sum_{j=k}^{m}\frac{1}{r}\left(\frac{-s}{r}\right)^{j-k}\frac{f_{j+1}^2}{f_k f_{k+1}}a_{nj}\right]y_k \text{ for all } n \in \mathbb{N}.$$

By passing to limit as $m \to \infty$ it is seen that $Ty = Ax$ and this illustrates that $A \in \left(\widehat{c}^{f(r,s)} : X\right)$.

In fact, this is exactly what we want to prove. □

Theorem 3.2. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $R = (r_{nk})$ are connected with the relation $r_{nk} = \bar{a}_{nk}$ for all $k, n \in \mathbb{N}$ and X be given sequence space. Then, $A \in \left(X : \widehat{c}^{f(r,s)}\right)$ if and only if $R \in (X : \widehat{c})$.*

Proof. Let us take any sequence $x = (x_k) \in X$ and we deal with the following equalities

$$\begin{aligned}\left\{\widehat{F}\left(Ax\right)\right\}_n &= r\frac{f_n}{f_{n+1}}\left(Ax\right)_n + s\frac{f_{n+1}}{f_n}\left(Ax\right)_{n-1} \\ &= r\frac{f_n}{f_{n+1}}\sum_k a_{nk}x_k + s\frac{f_{n+1}}{f_n}\sum_k a_{n-1,k}x_k \\ &= \sum_k\left(r\frac{f_n}{f_{n+1}}a_{nk} + s\frac{f_{n+1}}{f_n}a_{n-1,k}\right)x_k = \left(Rx\right)_n\end{aligned}$$

for all $n \in \mathbb{N}$, from elementary calculus. By passing to generalized limit in newly obtained formula. It is not hard to say that $Ax \in \widehat{c}^{f(r,s)}$ if and only if $Rx \in \widehat{c}$. This completes the proof. $\square$

Now, we give the following conditions:

$$\sup_{n\in\mathbb{N}}\sum_k |\Delta a_{nk}| < \infty, \tag{3.2}$$

$$\lim_{k\to\infty} a_{nk} = 0 \text{ for each fixed } n \in \mathbb{N}, \tag{3.3}$$

$$\widehat{c} - \lim a_{nk} = \alpha_k \text{ exists for each fixed } k \in \mathbb{N}, \tag{3.4}$$

$$\lim_{m\to\infty}\sum_k |a\left(n,k,m\right) - \alpha_k| = 0 \text{ uniformly in } n, \tag{3.5}$$

$$\widehat{c} - \lim\sum_k a_{nk} = \alpha, \tag{3.6}$$

$$\lim_{m\to\infty}\sum_k |\Delta\left[a\left(n,k,m\right) - \alpha_k\right]| = 0 \text{ uniformly in } n, \tag{3.7}$$

$$\lim_{q\to\infty}\sum_k \frac{1}{q+1}\left|\sum_{i=0}^{q}\Delta\left[a\left(n+i,k\right) - \alpha_k\right]\right| = 0 \text{ uniformly in } n, \tag{3.8}$$

$$\sup_{n\in\mathbb{N}}\sum_k |a\left(n,k\right)| < \infty, \tag{3.9}$$

$$\sum_n a_{nk} = \alpha_k \text{ for each fixed } n \in \mathbb{N}, \tag{3.10}$$

$$\sum_n\sum_k a_{nk} = \alpha, \tag{3.11}$$

$$\lim_{n\to\infty}\sum_k |\Delta\left[a\left(n,k\right) - \alpha_k\right]| = 0. \tag{3.12}$$

Since it will help very much in the implementation process, let us state the previously obtained results related to almost convergence as a Lemma.

Lemma 3.3. *[11] Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*
(i) $A = (a_{nk}) \in (\widehat{c} : \ell_\infty)$ if and only if (2.6) holds.
(ii) $A = (a_{nk}) \in (\ell_\infty : \widehat{c})$ if and only if (2.6), (3.4) and (3.5) hold.
(iii) $A = (a_{nk}) \in (\widehat{c} : \widehat{c})$ if and only if (2.6), (3.4), (3.6) and (3.7) hold.
(iv) $A = (a_{nk}) \in (c : \widehat{c})$ if and only if (2.6), (3.4) and (3.6) hold.
(v) $A = (a_{nk}) \in (bs : \widehat{c})$ if and only if (3.2), (3.3), (3.4) and (3.8) hold.
(vi) $A = (a_{nk}) \in (cs : \widehat{c})$ if and only if (3.2) and (3.4) hold.
(vii) $A = (a_{nk}) \in (\widehat{c} : cs)$ if and only if $(3.9) - (3.12)$ hold.

Later, using Theorems 3.1 and 3.2 with together with Lemmas 2.4 and 3.3 will results in the following corollaries.

Corollary 3.4. *The following statements hold:*

(i) $A = (a_{nk}) \in \left(\widehat{c}^{f(r,s)} : \ell_\infty\right)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{c}^{f(r,s)}\right\}^\beta$ *for all* $n \in \mathbb{N}$ *and* (2.6) *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

(ii) $A = (a_{nk}) \in \left(\widehat{c}^{f(r,s)} : c\right)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{c}^{f(r,s)}\right\}^\beta$ *for all* $n \in \mathbb{N}$ *and* $(2.3)-(2.6)$ *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

(iii) $A = (a_{nk}) \in \left(\widehat{c}^{f(r,s)} : \widehat{c}\right)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{c}^{f(r,s)}\right\}^\beta$ *for all* $n \in \mathbb{N}$ *and* (2.6), (3.4), (3.6) *and* (3.7) *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

(iv) $A = (a_{nk}) \in \left(\widehat{c}^{f(r,s)} : bs\right)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{c}^{f(r,s)}\right\}^\beta$ *for all* $n \in \mathbb{N}$ *and* (3.9) *holds.*

(v) $A = (a_{nk}) \in \left(\widehat{c}^{f(r,s)} : cs\right)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{\widehat{c}^{f(r,s)}\right\}^\beta$ *for all* $n \in \mathbb{N}$ *and* $(3.9)-(3.12)$ *hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 3.5. *The following statements hold:*

(i) $A = (a_{nk}) \in \left(\ell_\infty : \widehat{c}^{f(r,s)}\right)$ *if and only if* (2.6), (3.4) *and* (3.5) *hold with* $\bar{a}_{nk}$ *instead of* a_{nk}.

(ii) $A = (a_{nk}) \in \left(c : \widehat{c}^{f(r,s)}\right)$ *if and only if* (2.6), (3.4) *and* (3.6) *hold with* $\bar{a}_{nk}$ *instead of* a_{nk}.

(iii) $A = (a_{nk}) \in \left(\widehat{c} : \widehat{c}^{f(r,s)}\right)$ *if and only if* (2.6), (3.4), (3.6) *and* (3.7) *hold with* $\bar{a}_{nk}$ *instead of* a_{nk}.

(iv) $A = (a_{nk}) \in \left(bs : \widehat{c}^{f(r,s)}\right)$ *if and only if* (3.2), (3.3), (3.4) *and* (3.8) *hold with* $\bar{a}_{nk}$ *instead of* a_{nk}.

(v) $A = (a_{nk}) \in \left(cs : \widehat{c}^{f(r,s)}\right)$ *if and only if* (3.2) *and* (3.4) *hold with* $\bar{a}_{nk}$ *instead of* a_{nk}.

4 Core Theorems

Let $x = (x_k)$ be a sequence in $\mathbb{C}$, the set of all complex numbers, and R_k be the least convex closed region of complex plane containing $x_k, x_{k+1}, x_{k+2}, \ldots$. The Knopp Core (or $\mathcal{K}-core$) of x is defined by the intersection of all R_k (k=1,2,...), (see [37], pp.137). In [38], it is shown that

$$\mathcal{K}-core(x) = \bigcap_{z\in\mathbb{C}} B_x(z)$$

for any bounded sequence x, where $B_x(z) = \left\{w \in \mathbb{C} : |w-z| \le \limsup_k |x_k - z|\right\}$.
Let E be a subset of $\mathbb{N}$. The natural density δ of E is defined by

$$\delta(E) = \lim_n \frac{1}{n} |\{k \le n : k \in E\}|$$

where $|\{k \le n : k \in E\}|$ denotes the number of elements of E not exceeding n. A sequence $x = (x_k)$ is said to be statistically convergent to a number l, if $\delta(\{k : |x_k - l| \ge \varepsilon\}) = 0$ for every ε. In this case we write $st-\lim x = l$, [39]. By st we denote the space of all statistically convergent sequences.

In [40], the notion of the statistical core (or $st-core$) of a complex valued sequence has been introduced by Fridy and Orhan and it is shown for a statistically bounded sequence x that

$$st-core(x) = \bigcap_{z\in\mathbb{C}} C_x(z),$$

where $C_x(z) = \{w \in \mathbb{C} : |w - z| \leq st - \limsup_k |x_k - z|\}$. The core theorems have been studied by many authors. For instance see [41, 42, 43, 44, 45, 46] and the others.
Using the convergence domain of the matrix $\widehat{F}(r,s) = (f_{nk}(r,s))$, the new sequence spaces $c_0(\widehat{F}(r,s))$ and $c(\widehat{F}(r,s))$ have been constructed and their some properties have been investigated by Candan [29]. In this section we will consider the sequences with complex entries and by $\ell_\infty(\mathbb{C})$ denote the space of all bounded complex valued sequences.

Following Knopp, a core theorem is characterized a class of matrices for which the core of the transformed sequence is included by the core of the original sequence. For example Knopp Core Theorem [37, p. 138] states that $\mathcal{K} - core(Ax) \subseteq \mathcal{K} - core(x)$ for all real valued sequences x whenever A is a positive matrix in the class $(c : c)_{reg}$.

Here, we will define $B_{\widehat{F}(r,s)}$-core of a complex valued sequence and characterize the class of matrices to yield $B_{\widehat{F}(r,s)} - core(Ax) \subseteq \mathcal{K} - core(x)$, $\mathcal{K} - core(Ax) \subseteq B_{\widehat{F}(r,s)} - core(x)$, $B_{\widehat{F}(r,s)} - core(Ax) \subseteq B_{\widehat{F}(r,s)} - core(x)$ and $B_{\widehat{F}(r,s)} - core(Ax) \subseteq st - core(x)$ for all $x \in \ell_\infty$.

Now, let us write

$$t_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m}\left(r\frac{f_{n+j}}{f_{n+1+j}}x_{n+j} + s\frac{f_{n+1+j}}{f_{n+j}}x_{n-1+j}\right).$$

Then, we can define $B_{\widehat{F}(r,s)} - core$ of a complex sequence as follows:

Definition 4.1. *Let H_n be the least closed convex hull containing $t_{mn}(x)$, $t_{m+1,n}(x)$, $t_{m+2,n}(x), \ldots$. Then, $B_{\widehat{F}(r,s)} - core$ of x is the intersection of all H_n, i.e.,*

$$B_{\widehat{F}(r,s)} - core(x) = \bigcap_{n=1}^{\infty} H_n.$$

Note that, actually, we define $B_{\widehat{F}(r,s)} - core$ of x by the $\mathcal{K} - core$ of the sequence $(t_{mn}(x))$. Hence, we can construct the following theorem which is an analogue of $\mathcal{K} - core$, [38].

Theorem 4.2. *For any $z \in \mathbb{C}$, let*

$$G_x(z) = \left\{\omega \in \mathbb{C} : |\omega - z| \leq \limsup_{m\to\infty}\sup_{n\in\mathbb{N}} |t_{mn}(x) - z|\right\}.$$

Then, for any $x \in \ell_\infty$,

$$B_{\widehat{F}(r,s)} - core(x) = \bigcap_{z\in\mathbb{C}} G_x(z).$$

Now, we prove some lemmas which will be useful to the main results of this section. To do these, we need to characterize the classes $(c : \widehat{c}^{f(r,s)})_{reg}$ and $(st \cap \ell_\infty : \widehat{c}^{f(r,s)})_{reg}$. For brevity, in what follows we write $\widetilde{a}(m,n,k)$ in place of

$$\frac{1}{m+1}\sum_{j=0}^{m}\left(r\frac{f_{n+j}}{f_{n+1+j}}a_{n+j,k} + s\frac{f_{n+1+j}}{f_{n+j}}a_{n-1+j,k}\right)$$

for all $m, n, k \in \mathbb{N}$.

Lemma 4.3. $A \in (\ell_\infty : \widehat{c}^{f(r,s)})$ *if and only if*

$$\|A\| = \sup_{m,n}\sum_k |\tilde{a}(m,n,k)| < \infty, \tag{4.1}$$

$$\lim_{m\to\infty} \tilde{a}(m,n,k) = \alpha_k \text{ for each } k, \tag{4.2}$$

$$\lim_{m\to\infty}\sum_k |\tilde{a}(m,n,k)-\alpha_k| = 0, \text{ uniformly in } n. \tag{4.3}$$

Lemma 4.4. *$A \in (c : \widehat{c}^{f(r,s)})_{reg}$ if and only if (4.1) and (4.2) of the Lemma 4.3 hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$ and*

$$\lim_{m\to\infty}\sum_k \tilde{a}(m,n,k) = 1 \text{ uniformly in } n. \tag{4.5}$$

Lemma 4.5. *$A \in (st \cap \ell_\infty : \widehat{c}^{f(r,s)})_{reg}$ if and only if $A \in (c : \widehat{c}^{f(r,s)})_{reg}$ and*

$$\lim_{m\to\infty}\sum_{k\in E} |\tilde{a}(m,n,k)| = 0 \text{ uniformly in } n \tag{4.6}$$

for every $E \subset \mathbb{N}$ with natural density zero.

Proof. Let $A \in (st \cap \ell_\infty : \widehat{c}^{f(r,s)})_{reg}$. Then $A \in (c : \widehat{c}^{f(r,s)})_{reg}$ immediately follows from the fact that $c \subset st \cap \ell_\infty$. Now, define a sequence $t = (t_k)$ for $x \in \ell_\infty$ as $t_k = \begin{cases} x_k, & k \in E \\ 0, & k \notin E. \end{cases}$ where E any subset of $\mathbb{N}$ with $\delta(E) = 0$. Then, $st - \lim t_n = 0$ and $t \in st_0$, so we have $At \in \widehat{c_0}^{f(r,s)}$. On the other hand, since $(At)_n = \sum_{k\in E} a_{nk}t_k$, the matrix $B = (b_{nk})$ defined by $b_{nk} = \begin{cases} a_{nk}, & k \in E \\ 0, & k \notin E. \end{cases}$ for all n, must belong to the class $(\ell_\infty : \widehat{c_0}^{f(r,s)})$. Hence, the necessity of (4.6) follows from Lemma 4.3.

Conversely, let $x \in st \cap \ell_\infty$ with $st - \lim x = l$. Then, the set E defined by $E = \{k : |x_k - l| \geq \varepsilon\}$ has density zero and $|x_k - l| \leq \varepsilon$ if $k \notin E$. Now, we can write

$$\sum_k \tilde{a}(m,n,k)x_k = \sum_k \tilde{a}(m,n,k)(x_k - l) + l\sum_k \tilde{a}(m,n,k). \tag{4.7}$$

Since

$$\left|\sum_k \tilde{a}(m,n,k)(x_k - l)\right| \leq \|x\| \sum_{k\in E} |\tilde{a}(m,n,k)| + \varepsilon \cdot \|A\|,$$

letting $m \to \infty$ in (4.7) and using (4.5) with (4.6), we have

$$\lim_{m\to\infty}\sum_k \tilde{a}(m,n,k)x_k = l.$$

This implies that $A \in (st \cap \ell_\infty : \widehat{c}^{f(r,s)})_{reg}$ and the proof is completed. □

Now, we may give some inclusion theorems. Firstly, we need a lemma.

Lemma 4.6. *[47, Corollary 12] Let $\mathcal{A} = \{a_{mk}(n)\}$ defined by $a_{mk}(n) = \tilde{a}(m,n,k)$ for all $m, n, k \in \mathbb{N}$ be a matrix satisfying $||\mathcal{A}|| = ||a_{mk}(n)|| < \infty$ and $\lim_m \sup_{n\in\mathbb{N}} |a_{mk}(n)| = 0$. Then, there exists an $y \in \ell_\infty$ with $\|y\| \leq 1$ such that*

$$\limsup_{m\to\infty}\sup_{n\in\mathbb{N}}\sum_k \tilde{a}(m,n,k)y_k = \limsup_{m\to\infty}\sup_{n\in\mathbb{N}}\sum_k |\tilde{a}(m,n,k)|.$$

Theorem 4.7. *$B_{\widehat{F}(r,s)} - core(Ax) \subseteq \mathcal{K} - core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (c : \widehat{c}^{f(r,s)})_{reg}$ and*

$$\limsup_{m\to\infty}\sup_{n\in\mathbb{N}}\sum_k |\tilde{a}(m,n,k)| = 1. \tag{4.8}$$

Proof. Let the $B_{\widehat{F}(r,s)} - core(Ax) \subseteq \mathcal{K} - core(x)$ and take $x \in c$ with $\lim x = \ell$. Then, since $\mathcal{K} - core(x) \subseteq \{\ell\}$, $B_{\widehat{F}(r,s)} - core(Ax) \subseteq \{\ell\}$. So, $\widehat{c}^{f(r,s)} - limAx = \ell$ which means that $A \in (c : \widehat{c}^{f(r,s)})_{reg}$.

Since $A \in (c : \widehat{c}^{f(r,s)})_{reg}$, the matrix $\mathcal{A} = \tilde{a}(m,n,k)$ is satisfy the conditions of Lemma 4.6. So, there exists a $y \in \ell_\infty$ with $\|y\| \leq 1$ such that

$$\left\{\omega \in \mathbb{C} : |\omega| \leq \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \sum_k \tilde{a}(m,n,k)y_k\right\} = \left\{\omega \in \mathbb{C} : |\omega| \leq \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \sum_k |\tilde{a}(m,n,k)|\right\}.$$

On the other hand, since $\mathcal{K} - core(y) \subseteq A_1(0)$, by the hypothesis

$$\left\{\omega \in \mathbb{C} : |\omega| \leq \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \sum_k |\tilde{a}(m,n,k)|\right\} \subseteq A_1(0) = \{\omega \in \mathbb{C} : |\omega| \leq 1\}$$

which implies (4.8).

Conversely, let $\omega \in B_{\widehat{F}(r,s)} - core(Ax)$. Then, for any given $z \in \mathbb{C}$, we can write

$$\begin{aligned} |\omega - z| &\leq \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} |t_{mn}(Ax) - z| \\ &= \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \left| z - \sum_k \tilde{a}(m,n,k)x_k \right| \\ &\leq \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \left| \sum_k \tilde{a}(m,n,k)(z - x_k) \right| \\ &+ \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} |z| \left| 1 - \sum_k \tilde{a}(m,n,k) \right| \\ &= \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \left| \sum_k \tilde{a}(m,n,k)(z - x_k) \right|. \end{aligned} \tag{4.9}$$

Now, let $L(x) = \limsup_k |x_k - z|$. Then, for any $\varepsilon > 0$, $|x_k - z| \leq L(x) + \varepsilon$ whenever $k \geq k_0$. Hence, one can write that

$$\begin{aligned} \left| \sum_k \tilde{a}(m,n,k)(z - x_k) \right| &= \left| \sum_{k<k_0} \tilde{a}(m,n,k)(z - x_k) + \sum_{k\geq k_0} \tilde{a}(m,n,k)(z - x_k) \right| \\ &\leq \sup_k |z - x_k| \sum_{k<k_0} |\tilde{a}(m,n,k)| \\ &+ [L(x) + \varepsilon] \sum_{k\geq k_0} |\tilde{a}(m,n,k)| \\ &\leq \sup_k |z - x_k| \sum_{k<k_0} |\tilde{a}(m,n,k)| \\ &+ [L(x) + \varepsilon] \sum_{k\geq k_0} |\tilde{a}(m,n,k)|. \end{aligned} \tag{4.10}$$

Therefore, applying $\limsup_{m\to\infty} \sup_{n\in\mathbb{N}}$ under the light of the hypothesis and combining (4.9) with (4.10), we have

$$|\omega - z| \leq \limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \left| \sum_k \tilde{a}(m,n,k)(z - x_k) \right| \leq L(x) + \varepsilon$$

which means that $\omega \in \mathcal{K} - core(x)$. This completes the proof. $\square$

The proof of the following two theorems are entirely analogous to the Theorem 4.7. So, we omit the detail.

Theorem 4.8. *$\mathcal{K} - core(Ax) \subseteq B_{\widehat{F}(r,s)} - core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (\widehat{c}^{f(r,s)} : c)_{reg}$ and (4.8) holds.*

Theorem 4.9. *$B_{\widehat{F}(r,s)} - core(Ax) \subseteq B_{\widehat{F}(r,s)} - core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (\widehat{c}^{f(r,s)} : \widehat{c}^{f(r,s)})_{reg}$ and (4.8) holds.*

Theorem 4.10. *$B_{\widehat{F}(r,s)} - core(Ax) \subseteq st - core(x)$ for all $x \in \ell_\infty$ if and only if $A \in (st \cap \ell_\infty : \widehat{c}^{f(r,s)})_{reg}$ and (4.8) holds.*

Proof. Firstly, we assume that $B_{\widehat{F}(r,s)} - core(Ax) \subseteq st - core(x)$ for all $x \in \ell_\infty$. By taking $x \in st \cap \ell_\infty$, one can see that $A \in (st \cap \ell_\infty : \widehat{c}^{f(r,s)})_{reg}$. Also, since $st - core(x) \subseteq \mathcal{K} - core(x)$ [42] for any x, the necessity of the condition (4.8) follows from Theorem 4.7.
Conversely, let $A \in (st \cap \ell_\infty : \widehat{c}^{f(r,s)})_{reg}$ and (4.8) holds and take $\omega \in B_{\widehat{F}(r,s)} - core(Ax)$. Then we can write again equality (4.9). Now, let $\beta = st - \limsup |z - x_k|$. If we write $E = \{k : |x_k - z| \geq \beta + \varepsilon\}$, then $\delta(E) = 0$ and $|z - x_k| \leq \beta + \varepsilon$ whenever $k \notin E$. Hence we have

$$\begin{aligned}
\left|\sum_k \tilde{a}(m,n,k)(z - x_k)\right| &= \left|\sum_{k\in E} \tilde{a}(m,n,k)(z - x_k) + \sum_{k\notin E} \tilde{a}(m,n,k)(z - x_k)\right| \\
&\leq |z - x_k| \sum_{k\in E} |\tilde{a}(m,n,k)| + \sum_{k\notin E} |\tilde{a}(m,n,k)||z - x_k| \\
&\leq |z - x_k| \sum_{k\in E} |\tilde{a}(m,n,k)| + [\beta + \varepsilon] \sum_{k\notin E} |\tilde{a}(m,n,k)|.
\end{aligned}$$

Thus, applying the operator $\limsup_{m\to\infty} \sup_{n\in\mathbb{N}}$ and using the hypothesis (4.8) with (4.6) , we obtain that

$$\limsup_{m\to\infty} \sup_{n\in\mathbb{N}} \left|\sum_k \tilde{a}(m,n,k)(z - x_k)\right| \leq \beta + \varepsilon. \tag{4.11}$$

Thus, (4.9) and (4.11) implies that $|\omega - z| \leq \beta + \varepsilon$. Since ε is arbitrary, this means $\omega \in st - core(x)$, which completes the proof. $\square$

Conclusion

In the current study, the sequence space $\widehat{c}^{f(r,s)}$ derived by generalized difference Fibonacci matrix in which $r, s \in \mathbb{R} \setminus \{0\}$ has been introduced and compared with some well-known spaces defined previously. Then, it has been found out that the space has not a Schauder basis. In conclusion, the Fibonacci core of a complex-valued sequence has been presented and inclusion theorems with respect to Fibonacci core type are shown.

Acknowledgment

The authors would like to express their sincere gratitude to the referees for their valuable comments and suggestions about the paper , which led to a number of improvements in this paper.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Lorentz GG, A contribution to the theory of divergent sequences, Acta Math. 1948;80:167-190.

[2] Banach S. Théorie des opérations linéaires. Chelsea Publishing company. New York; 1978.

[3] Boos J. Classical and modern methods in summability, Oxford University Press Inc, New York; 2000.

[4] Candan M. Domain of the double sequential band matrix in the classical sequence spaces, J. Inequal. Appl. 2012;281.

[5] Candan M. A new sequence space isomorphic to the space $\ell(p)$ and compact operators, J. Math. Comput. Sci. 2014; 4(2):306-334.

[6] Candan M. Domain of the double sequential band matrix in the spaces of convergent and null sequences, Adv. Difference Equ. 2014;2014:163.

[7] Candan M, Güneş A. Paranormed sequence space of non-absolute type founded using generalized difference matrix, Proc. Nat. Acad. Sci. India Sect. A, in press.

[8] Candan M. Some new sequence spaces derived from the spaces of bounded, convergent and null sequences, Int. J. Mod. Math. Sci. 2014;12(2):74-87.

[9] Şengönül M, Kayaduman K. On the Riesz almost convergent sequence space, Abstr. Appl. Anal. 2012;2012:18. Article ID 691694. DOI:10.1155/2012/691694.

[10] Kayaduman K, Şengönül M. The spaces of Cesàro almost convergent sequences and core theorems, Acta Math. Sci. 2012;32B(6):2265-2278.

[11] Başar F, Kirişçi M. Almost convergence and generalized difference matrix, Comput. Math. Appl. 2011;61:602-611.

[12] Candan M. Almost convergence and double sequential band matrix, Acta. Math. Sci. 2014;34B(2):354-366.

[13] Kara EE, Elmaağaç K. On the u-difference almost sequence space and matrix transformations, Inter. J. Modern Math. Sci. in press.

[14] Mursaleen, Invariant means and some matrix transformations, Indian J. Pure Appl. Math. 1994;25(3):353-359.

[15] Mursaleen M, Savaş E, Aiyub M, Mohuiddine SA. Matrix transformations between the spaces of Cesàro sequences and invariant means, Int. J. Math. Math. Sci. 2006;2006(8).Article ID 74319. DOI:10.1155/IJMMS/2006/74319.

[16] Başar F, Solakİ. Almost-coercive matrix transformations, Rend. Mat. Appl. 1991;(7) 11 (2) 249-256.

[17] Başar F, Çolak R. Almost-conservative matrix transformations, Turkish J. Math. 1989;13(3):91-100.

[18] Başar F. $f-$conservative matrix sequences, Tamkang J. Math. 1991;22(2):205-212.

[19] King JP, Almost summable sequences, Proc. Amer. Math. Soc. 1966;17:1219-1225.

[20] Qamaruddin, Mohuiddine SA. Almost convergence and some matrix transformations, Filomat. 2007;21(2):261-266.

[21] Nanda S. Matrix transformations and almost boundedness, Glasnik Mat. 1979;34(14):99-107.

[22] Gupkari SA. Some new sequence spaces and almost convergence, Filomat. 2008;22(2):59-64.

[23] Das G, Kuttner B, Nanda S. On absolute almost convergence, J. Math. Anal. Appl. 1991;161(1):50-56.

[24] Koshy T. Fibonacci and Lucas Numbers with Applications. Wiley; 2001.

[25] Kara EE. Some topological and geometrical properties of new Banach sequence spaces, J. Inequal. Appl.2013;2013(38):15.

[26] Kara EE, Başarır M, Mursaleen M. Compact operators on the Fibonacci difference sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$, 1st International Eurasian Conf. on Math. Sci. and Appl, Prishtine-Kosovo, September 3-7, 2012.

[27] Başarır M, Başar F, Kara EE. On the spaces of Fibonacci difference null and convergent sequences, arXiv:1309.0150.

[28] Kara EE, Demiriz S. Some new paranormed Fibonacci difference sequence spaces, 2st International Eurasian Conf. on Math. Sci. and Appl, Sarajevo-Bosnia and Herzegouina, August 26-29, 2013.

[29] Candan M. A new approach on the spaces of generalized Fibonacci difference null and convergent sequences, under communication.

[30] Candan M, Kara EE. A study on topological and geometrical characteristics of new Banach sequence spaces, under communication.

[31] Demiriz S, Kara EE, Başarır M. On the Fibonacci almost sequence space and Fibonacci core, Kyungpook Math. J. in press.

[32] Duran JP. Infinite matrices and almost convergence, Math. Z. 1972;128:75-83.

[33] Başar F. Summability theory and its applications, Bentham Science Publishers, e-books, Monographs, İstanbul; 2012.

[34] Miller HI, Orhan C, On almost convergent and statistically convergent subsequences, Acta Math. Hung. 2001;93:135-151.

[35] Jarrah AM, Malkowsky E. BK spaces, bases and linear operators, Ren. Circ. Mat. Palermo II. 1990;52:177-191.

[36] Sıddıqi JA. Infinite matrices summing every almost periodic sequences, Pacific. J. Math. 1971;39 (1):235-251.

[37] Cooke RG. Infinite matrices and sequence spaces, Mcmillan, New York; 1950.

[38] Shcherbakov AA. Kernels of sequences of complex numbers and their regular transformations, Math. Notes. 1977;22:948-953.

[39] Steinhaus H. Quality control by sampling, Collog. Math. 1951;2:98-108.

[40] Fridy JA, Orhan C. Statistical core theorems, J. Math. Anal. Appl. 1997;208:520-527.

[41] Allen HS. T-transformations which leave the core of every bounded sequence invariant, J. London Math. Soc. 1944;19:42-46.

[42] Connor J, Fridy JA, Orhan C. Core equality results for sequences, J. Math. Anal. Appl. 2006;321:515-523.

[43] Çakan C, Çoşkun H. Some new inequalities related to the invariant means and uniformly bounded function sequences, Appl. Math. Lett. 2007;20(6):605-609.

[44] Çoşkun H, Çakan C. A class of statistical and σ-conservative matrices, Czechoslovak Math. J. 2005;55(3):791-801.

[45] Çoşkun H, Çakan C, Mursaleen. On the statistical and σ-cores, Studia Math. 2003;154(1):29-35.

[46] Kayaduman K, Furkan H. Infinite matrices and $\sigma^{(A)}$-core, Demonstratio Math. 2006;39:531-538.

[47] Simons S. Banach limits, infinite matrices and sublinear functionals, J. Math. Anal. Appl. 1969;26:640-655.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=933&id=6&aid=8084