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Demiclosed Principle and $\triangle-$ Convergence of Fixed Points for Total Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract

In this paper, we prove the existence of fixed points and demiclosed principle for total asymptotically nonexpansive mappings in hyperbolic spaces. As a consequence, we obtain a $\triangle-$convergence theorem for such mappings in hyperbolic spaces. Our results improve and extend some results in the literature.

Keywords: Total asymptotically nonexpansive mappings, hyperbolic spaces, demiclosed principle.
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1 Introduction

In this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [1]. (X, d, W) is called a hyperbolic space if (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is a function satisfying

(I) $\forall x, y, z \in X,\ \forall \lambda \in [0, 1],\ d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y);$

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(II) $\forall x, y \in X,\ \forall \lambda_1, \lambda_2 \in [0,1],\ d(W(x,y,\lambda_1), W(x,y,\lambda_2)) = |\lambda_1 - \lambda_2| \cdot d(x,y);$

(III) $\forall x, y \in X,\ \forall \lambda \in [0,1],\ W(x,y,\lambda) = W(y,x,(1-\lambda));$

(IV) $\forall x, y, z, w \in X,\ \forall \lambda \in [0,1],\ d(W(x,z,\lambda), W(y,w,\lambda)) \leq (1-\lambda)d(x,y) + \lambda d(z,w).$

If a metric space satisfies only (I), it coincides with the convex metric space introduced by Takahashi [2]. The concept of hyperbolic space in [1] is more restrictive than the hyperbolic type introduced by Goebel [3] since (I)–(III) are equivalent to (X, d, W) being a space of hyperbolic type in [3]. But it is slightly more general than the hyperbolic space defined by Reich [4] (see [1]). This class of metric spaces in [1] covers all normed linear spaces, the Hilbert ball with the hyperbolic metric (see [5]), Cartesian products of Hilbert balls, Hadamard manifolds (see [4, 6]), $\mathbb{R}$-trees in the sense of Tits and CAT(0) spaces in the sense of Gromov (see [7]). A thorough discussion of hyperbolic spaces and a detailed treatment of examples can be found in [1] (see also [3-5]).

A hyperbolic space X is *uniformly convex* [8] if for $u, x, y \in X,\ r > 0$ and $\varepsilon \in (0,2]$ there exists a $\delta \in (0,1]$ such that

$$d\left(W(x,y,\frac{1}{2}), u\right) \leq (1-\delta)r,$$

provided that $d(x,u) \leq r,\ d(y,u) \leq r$ and $d(x,y) \geq \varepsilon r$.

A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ is called *modulus of uniform convexity* if $\delta = \eta(r,\epsilon)$ for given $r > 0$. Moreover, η is *monotone* if it decreases with r (for a fixed ϵ), that is,

$$\eta(r_2,\epsilon) \leq \eta(r_1,\epsilon),\ \forall r_2 \geq r_1 > 0.$$

A subset C of a hyperbolic space X is *convex* if $W(x,y,\lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$. For any $x \in X,\ r > 0$, the open (closed) ball with center x and radius r is denoted by $U(x,r)$ (respectively $\overline{U}(x,r)$).

Let (X,d) be a metric space and let C be a nonempty subset of X. Recall that a mapping $T : C \to C$ is said to be a $(\{\nu_n\}, \{\mu_n\}, \zeta)$-*total asymptotically nonexpansive mapping* if there exist nonnegative sequences $\{\nu_n\},\ \{\mu_n\}$ with $\nu_n \to 0,\ \mu_n \to 0$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x,y) + \nu_n \zeta(d(x,y)) + \mu_n,\ \forall n \geq 1,\ x, y \in C. \tag{1.1}$$

It is well known that each nonexpansive mapping is an asymptotically nonexpansive mapping and each asymptotically nonexpansive mapping is a $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive mapping.

$T : C \to C$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x,y),\ \forall n \geq 1,\ x, y \in C.$$

Recently, Kohlenbach and Leustean [9] proved the existence of fixed points and demiclosed principle for asymptotically nonexpansive mappings in hyperbolic spaces. Later, Zhang and Cui [10] obtained the existence of fixed points and demiclosed principle for mappings of asymptotically nonexpansive type in hyperbolic spaces. Motivated by [9] and [10], our purpose of this paper is to discuss the existence of fixed points and demiclosed principle for total asymptotically nonexpansive mappings in hyperbolic spaces.

2 Preliminaries

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, we define

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ *of* $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The *asymptotic radius* $r_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is given by

$$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The *asymptotic center* $A(\{x_n\})$ *of* $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

The *asymptotic center* $A_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}.$$

In 1976, Lim [11] introduced the concept of $\triangle-$ convergence in a general metric space. Recall that a sequence $\{x_n\}$ in X is said to $\triangle-$ *converge to* $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we call x the $\triangle-$ *limit of* $\{x_n\}$.

The following lemmas are important in our paper.

Lemma 2.1. *[9] Let* (X, d, W) *be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then the intersection of any decreasing sequence of nonempty bounded closed convex subsets of* X *is nonempty.*

Lemma 2.2. *[12,13] Let* (X, d, W) *be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity and let* C *be a nonempty closed convex subset of* X*. Then every bounded sequence* $\{x_n\}$ *in* X *has a unique asymptotic center with respect to* C*.*

Lemma 2.3. *[12] Let* (X, d, W) *be a uniformly convex hyperbolic space with monotone modulus of uniform convexity* η*. Let* $x \in X$ *and* $\{\alpha_n\}$ *be a sequence in* $[a, b]$ *for some* $a, b \in (0, 1)$*. If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that* $\limsup_{n\to\infty} d(x_n, x) \le c$*,* $\limsup_{n\to\infty} d(y_n, x) \le c$ *and* $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c$ *for some* $c \ge 0$*. Then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.4. *[14] Let* $\{a_n\}$*,* $\{b_n\}$ *and* $\{c_n\}$ *be sequences of nonnegative numbers such that*

$$a_{n+1} \le (1 + b_n)a_n + c_n, \ \ \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$*, then* $\lim_{n\to\infty} a_n$ *exists.*

3 Main Results

In this section, we prove our main theorems.

Theorem 3.1. *(Existence of fixed points for total asymptotically nonexpansive mappings in hyperbolic spaces) Let* (X, d, W) *be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity* η*. Let* C *be a nonempty bounded closed convex subset of* X*. Then every continuous total asymptotically nonexpansive mapping* $T : C \to C$ *has a fixed point.*

Proof. For any $y \in C$, let

$$B_y := \{b \in \mathbb{R}^+ : \text{there exist } x \in C \text{ and } k \ge 1 \text{ such that } d(T^i y, x) \le b \text{ for } i \ge k\}.$$

B_y is nonempty since $diam(C) \in B_y$. Define $\beta_y := \inf B_y$. For any $\theta > 0$, there exists $b_\theta \in B_y$ such that $b_\theta < \beta_y + \theta$. Then there exist $x \in C$ and $k \geq 1$ such that

$$d(T^i y, x) \leq b_\theta < \beta_y + \theta, \ \forall i \geq k. \tag{3.1}$$

It is easy to see that $\beta_y \geq 0$. We consider the following two cases:

$Case$ 1. $\beta_y = 0$. Let $\varepsilon > 0$ and apply (3.1) with $\theta = \frac{\varepsilon}{2}$. Then there exist $x \in C$ and $k \geq 1$ such that for all $i, j \geq k$

$$d(T^i y, T^j y) \leq d(T^i y, x) + d(T^j y, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which implies that $\{T^i y\}$ is a Cauchy sequence. Assume that $T^i y \to z$ as $i \to \infty$ for some $z \in C$. By the definition of T, we obtain

$$\begin{aligned} d(z, T^i z) &\leq d(z, T^{2i} y) + d(T^{2i} y, T^i z) \\ &= d(z, T^{2i} y) + d(T^i z, T^i T^i y) \\ &\leq d(z, T^{2i} y) + d(z, T^i y) + \nu_i \zeta(d(z, T^i y)) + \mu_i \to 0 \text{ as } i \to \infty. \end{aligned}$$

Thus $T^i z \to z$ as $i \to \infty$. By the continuity of T, we get

$$Tz = T(\lim_{i\to\infty} T^i z) = \lim_{i\to\infty} T^{i+1} z = z.$$

Hence, $z \in F(T)$.

$Case$ 2. $\beta_y > 0$. For any $n \geq 1$, let

$$C_n := \bigcup_{k\geq 1} \bigcap_{i\geq k} \overline{U}\left(T^i y, \beta_y + \frac{1}{n}\right), \ \ D_n := \overline{C_n} \bigcap C.$$

Taking $\theta = \frac{1}{n}$ in (3.1), there exist $x \in C, \ k \geq 1$ such that $x \in \bigcap_{i\geq k} \overline{U}(T^i y, \beta_y + \frac{1}{n})$. Thus $\{D_n\}$ is a decreasing sequence of nonempty bounded closed convex subsets of X. By Lemma 1, we have

$$D := \bigcap_{n\geq 1} D_n \neq \emptyset.$$

For any $x \in D$ and $\theta > 0$, let $N \geq 1$ be such that $\frac{2}{N} \leq \theta$. It follows that $x \in \overline{C_N}$ and there exists a sequence $\{x_n^N\} \subset C_N$ such that $\lim_{n\to\infty} x_n^N = x$. Let $P \geq 1$ be such that $d(x, x_n^N) \leq \frac{1}{N}$ for all $n \geq P$ and let $K \geq 1$ be such that $x_P^N \in \bigcap_{i\geq K} \overline{U}(T^i y, \beta_y + \frac{1}{N})$. Then for all $i \geq K$, we have

$$d(T^i y, x) \leq d(T^i y, x_P^N) + d(x_P^N, x) \leq \beta_y + \frac{1}{N} + \frac{1}{N} \leq \beta_y + \theta. \tag{3.2}$$

Now we are in the position to prove that any point of D is a fixed point of T. Let $x \in D$ and assume by contradiction that $Tx \neq x$. Then $\{T^i x\}$ does not converge to x as $i \to \infty$ and so we can find $\varepsilon > 0$, for any $m_0 \geq 1$, there exists $m \geq m_0$ such that

$$d(T^m x, x) \geq \varepsilon. \tag{3.3}$$

Without loss of generality, we assume that $\varepsilon \in (0, 2]$. Then $\frac{\varepsilon}{\beta_y+1} \in (0, 2]$ and there exists $\theta_y \in (0, 1]$ such that

$$1 - \eta\left(\beta_y + 1, \frac{\varepsilon}{\beta_y + 1}\right) \leq \frac{\beta_y - \theta_y}{\beta_y + \theta_y}.$$

Taking $\theta = \frac{\theta_y}{2}$ in (3.2), there exists $K \geq 1$ such that

$$d(T^i y, x) \leq \beta_y + \frac{\theta_y}{2},\ \forall i \geq K. \tag{3.4}$$

By the definition of T, there exists M_0 such that if $i \geq M_0$, then we have

$$\begin{aligned} d(T^i x, T^i z) &\leq d(x,z) + \nu_i \zeta(d(x,z)) + \mu_i \\ &\leq d(x,z) + \frac{\theta_y}{2},\ \forall x, z \in C. \end{aligned} \tag{3.5}$$

By (3.3) with $m_0 = M_0$, there exists $M \geq M_0$ such that

$$d(T^M x, x) \geq \varepsilon. \tag{3.6}$$

Let $i \geq 1$ be such that $i \geq M + K$. It follows from (3.4), (3.5) and (3.6) that

$$d(x, T^i y) \leq \beta_y + \frac{\theta_y}{2} < \beta_y + \theta_y;$$

$$\begin{aligned} d(T^M x, T^i y) &= d(T^M x, T^M T^{i-M} y) \\ &\leq d(x, T^{i-M} y) + \frac{\theta_y}{2} \\ &\leq \beta_y + \theta_y; \end{aligned}$$

$$d(T^M x, x) \geq \varepsilon = \frac{\varepsilon}{\beta_y + \theta_y} \cdot (\beta_y + \theta_y) \geq \frac{\varepsilon}{\beta_y + 1} \cdot (\beta_y + \theta_y).$$

It follows from X is uniformly convex and η is monotone that

$$\begin{aligned} d(W(x, T^M x, \frac{1}{2}), T^i y) &\leq \left[1 - \eta\left(\beta_y + \theta_y, \frac{\varepsilon}{\beta_y + 1}\right)\right](\beta_y + \theta_y) \\ &\leq \left[1 - \eta\left(\beta_y + 1, \frac{\varepsilon}{\beta_y + 1}\right)\right](\beta_y + \theta_y) \\ &\leq \frac{\beta_y - \theta_y}{\beta_y + \theta_y} \cdot (\beta_y + \theta_y) \\ &= \beta_y - \theta_y. \end{aligned}$$

Hence, there exist $k := M + K$ and $z := W(x, T^M x, \frac{1}{2}) \in C$ such that for all $i \geq k$, $d(z, T^i y) \leq \beta_y - \theta_y$. It implies that $\beta_y - \theta_y \in B_y$, which contradicts with $\beta_y = \inf B_y$. Thus, $x \in F(T)$. ☐

It is well known that one of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's *demiclosed principle* [15] which states that X is a uniformly convex Banach space, C is a nonempty closed convex subset of X, and $T : C \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at 0, i.e., for any sequence $\{x_n\}$ in C if $x_n \to x$ weakly and $\|(I - T)x_n\| \to 0$, then $x = Tx$. In the following, we shall prove that a total asymptotically nonexpansive mapping in a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity is demiclosed. Let X be a hyperbolic space and let C be a nonempty closed convex subset of X. Let $\{x_n\}$ be a bounded sequence in C. In what follows, we denote it by

$$\{x_n\} \rightharpoonup \omega \text{ if and only if } \Phi(\omega) = \inf_{x \in C} \Phi(x),$$

where $\Phi(x) := \limsup_{n \to \infty} d(x_n, x)$.

Theorem 3.2. *(Demiclosed principle for total asymptotically nonexpansive mappings in hyperbolic spaces) Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone nodulus of uniform convexity η. Let C be a nonempty closed and convex subset of X. Let $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)-$ total asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{x_n\} \rightharpoonup p$. Then we have $T(p) = p$.*

Proof. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, by induction we can prove that

$$\lim_{n\to\infty} d(x_n, T^m x_n) = 0 \text{ for each } m \geq 1. \tag{3.7}$$

In fact, it is obvious that, the conclusion is true for $m = 1$. Suppose the conclusion holds for $m \geq 1$, now we prove that it is also true for $m + 1$. Indeed, since T is uniformly L-Lipschitzian, we have

$$\begin{aligned} d(x_n, T^{m+1}x_n) &\leq d(x_n, T^m x_n) + d(T^m x_n, T^{m+1}x_n) \\ &\leq d(x_n, T^m x_n) + Ld(x_n, Tx_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus (3.7) is proved. Now for each $x \in C$ and $m \geq 1$, from (3.7) we have

$$\Phi(x) := \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(T^m x_n, x). \tag{3.8}$$

In (3.8), taking $x = T^m p$, we get

$$\begin{aligned} \Phi(T^m p) &= \limsup_{n\to\infty} d(T^m x_n, T^m p) \\ &\leq \limsup_{n\to\infty} [d(x_n, p) + \nu_m \zeta(d(x_n, p)) + \mu_m]. \end{aligned}$$

Letting $m \to \infty$ and taking superior limit on the both sides, we have

$$\limsup_{m\to\infty} \Phi(T^m p) \leq \Phi(p). \tag{3.9}$$

We assume by contradiction that $Tp \neq p$. Then $\{T^m p\}$ does not converge to p as $m \to \infty$, so we can find $\varepsilon_0 > 0$, for any $k \geq 1$, there exists $m \geq k$ such that $d(T^m p, p) \geq \varepsilon_0$. We can assume $\varepsilon_0 \in (0, 2]$. Then $\frac{\varepsilon_0}{\Phi(p)+1} \in (0, 2]$ and there exists $\theta \in (0, 1]$ such that

$$1 - \eta\left(\Phi(p) + 1, \frac{\varepsilon_0}{\Phi(p) + 1}\right) \leq \frac{\Phi(p) - \theta}{\Phi(p) + \theta}. \tag{3.10}$$

By the definition of Φ and (3.9), there exist $N_1, M_1 \geq 1$ such that

$$d(p, x_n) \leq \Phi(p) + \theta, \ \forall n \geq N_1;$$

$$d(T^m p, x_n) \leq \Phi(p) + \theta, \ \forall n \geq N_1, \ m \geq M_1.$$

Besides, there exists $m \geq M_1$ such that

$$d(T^m p, p) \geq \varepsilon_0 = \frac{\varepsilon_0}{\Phi(p) + \theta} \cdot (\Phi(p) + \theta) \geq \frac{\varepsilon_0}{\Phi(p) + 1} \cdot (\Phi(p) + \theta).$$

Since X is uniformly convex and η is monotone, by (3.10) we get

$$\begin{aligned} d(W(p, T^m p, \frac{1}{2}), x_n) &\leq \left[1 - \eta\left(\Phi(p) + \theta, \frac{\varepsilon_0}{\Phi(p) + 1}\right)\right] \cdot (\Phi(p) + \theta) \\ &\leq \frac{\Phi(p) - \theta}{\Phi(p) + \theta} \cdot (\Phi(p) + \theta) \\ &= \Phi(p) - \theta. \end{aligned}$$

Hence $z := W(p, T^m p, \frac{1}{2}) \in C$ and $z \neq p$, which contradicts $\Phi(p) = \inf_{x\in C} \Phi(x)$. Thus $Tp = p$. □

Theorem 3.3. *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : C \to C$, $i = 1, 2$, be uniformly L-Lipschitzian and $(\{\nu_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mappings. Suppose that $F(T_1) \bigcap F(T_2) \neq \emptyset$. For arbitrarily chosen $x_1 \in C$, $\{x_n\}$ is defined as follows*

$$\begin{cases} x_{n+1} = W(x_n, T_1^n y_n, \alpha_n), \\ y_n = W(x_n, T_2^n x_n, \beta_n), \end{cases} \tag{3.11}$$

where the following conditions are satisfied:

- *(i)* $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty,\ i = 1, 2$;
- *(ii)* *there exist constants* $a, b \in (0, 1)$ *such that* $\{\alpha_n\} \subset [a, b]$;
- *(iii)* *there exists a constant* $M^* > 0$ *such that* $\zeta^{(i)}(r) \leq M^* r,\ r \geq 0, i = 1, 2.$

Then the sequence $\{x_n\}$ defined by (3.11) $\triangle-$converges to a common fixed point of T_1 and T_2.

Proof. Without loss of generality, we can assume that $T_i : C \to C$ both are $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive mappings, where $\nu_n = \max\{\nu_n^{(i)}, i = 1, 2\},\ \mu_n = \max\{\mu_n^{(i)}, i = 1, 2\}$ and $\zeta = \max\{\zeta^{(i)}, i = 1, 2\}$. It is easy to see that conditions (i) and (iii) are still satisfied. Now we divide our proof into three steps.

$Step$ 1. In the sequel, we shall show that

$$\lim_{n\to\infty} d(x_n, p) \text{ exists for each } p \in F(T_1) \cap F(T_2). \tag{3.12}$$

In fact, by conditions (1), (I) and (iii), one gets

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T_2^n x_n, \beta_n), p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(T_2^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n[d(x_n, p) + \nu_n \zeta(d(x_n, p)) + \mu_n] \\ &\leq (1 + \beta_n \nu_n M^*)d(x_n, p) + \beta_n \mu_n \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, T_1^n y_n, \alpha_n), p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(T_1^n y_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n[d(y_n, p) + \nu_n \zeta(d(y_n, p)) + \mu_n] \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n[(1 + \nu_n M^*)d(y_n, p) + \mu_n]. \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we have

$$d(x_{n+1}, p) \quad \leq \quad (1 + \sigma_n)d(x_n, p) + \xi_n,\ \forall n \geq 1, \tag{3.15}$$

where $\sigma_n = \alpha_n \nu_n M^*(1 + \beta_n + \beta_n \nu_n M^*)$ and $\xi_n = \alpha_n \mu_n(1 + \beta_n + \beta_n \nu_n M^*)$. Furthermore, using the condition (i), we get

$$\Sigma_{n=1}^{\infty} \nu_n < \infty \text{ and } \Sigma_{n=1}^{\infty} \mu_n < \infty, \tag{3.16}$$

a combination of (3.15), (3.16) and Lemma 4 shows that (3.12) is proved.

$Step$ 2. We claim that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0,\ i = 1, 2. \tag{3.17}$$

In fact, it follows from (3.12) that $\lim_{n\to\infty} d(x_n, p)$ exists for each given $p \in F(T_1) \cap F(T_2)$. Without loss of generality, we assume that

$$\lim_{n\to\infty} d(x_n, p) = c \geq 0. \tag{3.18}$$

By (3.13) and (3.18), one has

$$\liminf_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(y_n, p) \leq \lim_{n\to\infty} [(1 + \beta_n \nu_n M^*) d(x_n, p) + \beta_n \mu_n] = c. \tag{3.19}$$

Noting

$$\begin{aligned} d(T_1^n y_n, p) &= d(T_1^n y_n, T_1^n p) \\ &\leq d(y_n, p) + \nu_n \zeta(d(y_n, p)) + \mu_n \\ &\leq (1 + \nu_n M^*) d(y_n, p) + \mu_n, \ \forall n \geq 1, \end{aligned}$$

by (3.19) we obtain

$$\limsup_{n\to\infty} d(T_1^n y_n, p) \leq c. \tag{3.20}$$

Besides, by (3.15) we get

$$d(x_{n+1}, p) = d(W(x_n, T_1^n y_n, \alpha_n), p) \leq (1 + \sigma_n) d(x_n, p) + \xi_n,$$

which yields that

$$\lim_{n\to\infty} d(W(x_n, T_1^n y_n, \alpha_n), p) = c. \tag{3.21}$$

Now by (3.18), (3.20), (3.21) and Lemma 3, we have

$$\lim_{n\to\infty} d(x_n, T_1^n y_n) = 0. \tag{3.22}$$

On the other hand, we have

$$\begin{aligned} d(x_n, p) &\leq d(x_n, T_1^n y_n) + d(T_1^n y_n, p) \\ &\leq d(x_n, T_1^n y_n) + d(y_n, p) + \nu_n M^* d(y_n, p) + \mu_n \\ &= d(x_n, T_1^n y_n) + (1 + \nu_n M^*) d(y_n, p) + \mu_n, \end{aligned}$$

which implies that $\liminf_{n\to\infty} d(y_n, p) \geq c$. Combining with (3.19), it yields that

$$\lim_{n\to\infty} d(y_n, p) = c,$$

that is,

$$\lim_{n\to\infty} d(W(x_n, T_2^n x_n, \beta_n), p) = c.$$

By Lemma 3 we can also have that

$$\lim_{n\to\infty} d(x_n, T_2^n x_n) = 0. \tag{3.23}$$

By virtue of (3.23), we have

$$\begin{aligned} d(y_n, x_n) &= d(W(x_n, T_2^n x_n, \beta_n), x_n) \\ &\leq \beta_n d(T_2^n x_n, x_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.24}$$

Combining (3.22) and (3.24), one obtains

$$\begin{aligned} d(x_n, T_1^n x_n) &\le d(x_n, T_1^n y_n) + d(T_1^n y_n, T_1^n x_n) \\ &\le d(x_n, T_1^n y_n) + d(x_n, y_n) + \nu_n \zeta(d(x_n, y_n)) + \mu_n \\ &\le d(x_n, T_1^n y_n) + (1 + \nu_n M^*) d(x_n, y_n) \\ &\quad + \mu_n \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.25}$$

This jointly with (3.22) yields that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(x_n, T_1^n y_n, \alpha_n), x_n) \\ &\le \alpha_n d(T_1^n y_n, x_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.26}$$

Now by (3.23), (3.25) and (3.26), for each $i = 1, 2$, we get

$$\begin{aligned} d(x_n, T_i x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) + d(T_i^{n+1} x_{n+1}, T_i^{n+1} x_n) \\ &\quad + d(T_i^{n+1} x_n, T_i x_n) \\ &\le (1 + L) d(x_n, x_{n+1}) + d(x_{n+1}, T_i^{n+1} x_{n+1}) \\ &\quad + L d(T_i^n x_n, x_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, (3.17) is proved.

$Step$ 3. Now we are in a position to prove the $\triangle-$convergence of $\{x_n\}$. Since $\{x_n\}$ is bounded, by Lemma 2, it has a unique asymptotic center $A_C(\{x_n\}) = \{x^*\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A_C(\{u_n\}) = \{u\}$. Since $\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = 0$, it follow from Theorem 2 that $u \in F(T_1) \cap F(T_2)$. By the uniqueness of asymptotic centers, we get that $x^* = u$. It implies that x^* is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$, that is, $\{x_n\}$ $\triangle-$converges to $x^* \in F(T_1) \cap F(T_2)$. The proof is completed. □

Example 1. Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $C = [-1, 1]$. Define two mappings $T_1, T_2 : C \to C$ by

$$T_1(x) = \begin{cases} -2\sin\frac{x}{2}, & x \in [0, 1], \\ 2\sin\frac{x}{2}, & x \in [-1, 0), \end{cases}$$

and

$$T_2(x) = \begin{cases} x, & x \in [0, 1], \\ -x, & x \in [-1, 0). \end{cases}$$

It is proved in [16, Example 3.1] that both T_1 and T_2 are asymptotically nonexpansive mappings with $k_n = 1,\ \forall n \ge 1$. Therefore, they are total asymptotically nonexpansive mappings with $\nu_n = \mu_n = 0,\ \forall n \ge 1,\ \zeta(t) = t,\ \forall t \ge 0$. Moreover, they are uniformly L-Lipschitzian mappings with $L = 1$. $F(T_1) = \{0\}$ and $F(T_2) = \{x \in C : 0 \le x \le 1\}$. Let

$$\alpha_n = \frac{n}{2n+1},\ \beta_n = \frac{n}{3n+1}\ \forall n \ge 1.$$

Therefore, the conditions of Theorem 3 are fulfilled.

4 Conclusion

The major findings of this study are the proofs of the existence of fixed points and demiclosed principle for total asymptotically nonexpansive mappings in hyperbolic spaces.

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Competing Interests

The author declares that they have no competing interests.

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