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Solving k-Fractional Hilfer Differential Equations via Combined Fractional Integral Transform Methods

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Abstract

In this paper, we introduce k-fractional Hilfer derivative given in [1]. A combination of fractional Fourier transform method and Laplace transform method is adopted to solve Cauchy-type problems involving k-fractional Hilfer derivatives and an integral operator whose kernel contains k-Mittag-Leffler function similar to the one given in [2]. The solutions to these problems are obtained in terms of Mittag-Leffler function.

Keywords: k-fractional Hilfer derivative; fractional Fourier transform; Laplace transform; Mittag-Leffler function

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1 Introduction

Generally, for the past three decades, fractional calculus has been considered with great importance due to its various applications in fluid flow, control theory of dynamical systems, chemical physics, electrical networks,fractal heat transfer and so on, see [3]. The quest of getting accurate methods for solving resulted non-linear model involving fractional order is of almost concern of many researchers in this field today. In Caputo [4] and He [5], the approach used to account for the effects of changing flux is to embody the effects of memory which has to do with posing problem in terms of fractional calculus. Levy-flight type of transport is a well known diffusion process which is described by a fractional system.

Fractional Fourier transform (FRFT) was introduced in [6] to solve some classes of ordinary and partial differential equations found in quantum mechanics and some of its properties can be found in [7, 8, 9]. Several authors have introduced what we called k-fractional derivatives such as a k-Riemann Liouville fractional derivative in [10], a k-Riemann Liouville integral [11], the k-Wright function in [12] and a k-Mittag-Leffler function in [2]. Due to the properties involved in Hilfer fractional type of

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derivatives (introduced by Hilfer [1]) in the sense that it generalizes the Riemann-Liouville and Caputo fractional derivatives, a lot of studies have been done on it, including the existence and uniqueness of solutions to such differential equations involving Hilfer fractional derivatives see [13]. Apart from the classical methods introduced to solve this class of differential equations, some numerical and analytical methods have also been applied as in [5, 14, 15, 16, 17].

Following the method of solution analogous to that of [18] and [19], this paper is concerned with the solvability of Cauchy-type problems involving k-fractional Hilfer derivatives and an integral operator with a k-Mittag-Leffler function appearing in the Kernel using the combination of FRFT method and the classical Laplace transform method.

2 Preliminaries

In this section, we state some known results and some important definitions which will be used in the sequel.

Definition 2.1. *The Riemann-Liouville's (RL) fractional integral operator of order $\alpha \geq 0$, of a function $f \in L^1(a,b)$ is given as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \;\; t > 0, \;\; \alpha > 0, \tag{2.1}$$

where Γ is the Gamma function and $I_{0+}^{0} f(t) = f(t)$.

Definition 2.2. *The Riemann-Liouville's (RL) fractional derivative of order $0 < \alpha < 1$, of a function f is*

$$D_{0+}^{\alpha} f(t) = D I_{0+}^{1-\alpha} f(t). \tag{2.2}$$

provided the right-hand side exists where $D = d/dt$.

Definition 2.3. *The fractional derivative in the Caputo's sense is defined as [20],*

$${}^{C}D^{\alpha} f(t) = I^{n-\alpha} D^{n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{2.3}$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$.

Definition 2.4. *The fractional derivative of order $0 < \alpha < 1$ and the type $0 \leqslant \beta \leqslant 1$ with respect to t is defined as [19],*

$$D_{a^+}^{\alpha,\beta} f(t) = \left(I_{a^+}^{(1-\alpha)\beta} \frac{d}{dt} \left(I_{a^+}^{(1-\alpha)(1-\beta)} f \right) \right)(t), \tag{2.4}$$

for any function for which the right hand side expression exists.

Observe that (2.4) reduces to Riemann-Liouville fractional derivative (2.2) when $\beta = 0$ and also reduces to Caputo fractional derivative (2.3) when $\beta = 1$.

Lemma 2.1. *[20] Let $\alpha \geq 0$, $\beta \geq 0$ and $f \in CL(a,b)$. Then*

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(t) = I_{a^+}^{\alpha+\beta} f(t), \tag{2.5}$$

for all $t \in (a,b]$.

Lemma 2.2. *[20] Let $t \in (a,b]$. Then*

$$\left[I_{a+}^{\alpha} (t-a)^{\beta} \right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}, \qquad \alpha \geqslant 0, \quad \beta > 0. \tag{2.6}$$

Remark 1. *It is easy to see from the definitions given above that the Riemann-Liouville fractional derivative of a constant function is not equal to zero while that of Caputo fractional derivative of constant function is zero.*

Definition 2.5. *The k-gamma function Γ_k in the half-plane is defined as*

$$\Gamma_k(z) := \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt, \qquad Re(z) > 0, \quad t > 0. \tag{2.7}$$

When $k = 1$, we obtain the usual gamma function.

Lemma 2.3. *Let $0 < \alpha < 1$ and $k > 0$. Then*

$$\int_0^\infty t^{\frac{\alpha}{k}-1} e^{-st} dt = \frac{1}{s^{\frac{\alpha}{k}}} \Gamma\left(\frac{\alpha}{k}\right), \qquad s > 0. \tag{2.8}$$

Proof. Taking $m = st$ as a change of variable, we have the result. □

Lemma 2.4. *Let $0 < \alpha < 1$ and $k > 0$. Then*

$$\Gamma\left(\frac{\alpha}{k}\right) = k^{1-\frac{\alpha}{k}} \Gamma_k(\alpha), \qquad s > 0. \tag{2.9}$$

Proof. The result also follows by change of variable say $y = \frac{t^k}{k}$. □

Definition 2.6. *The Mittag-Leffler function with parameter α is given as*

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad Re(\alpha) > 0 \quad z \in \mathbb{C}. \tag{2.10}$$

Observe that $E_\alpha(z) = e^z$ for $\alpha = 1$.
It is defined on Cantor sets as given in ([21] with its graph) as

$$E_\alpha(x^\alpha) = \sum_{n=0}^\infty \frac{x^{\alpha n}}{\Gamma(\alpha n + 1)}, \qquad Re(\alpha) > 0. \tag{2.11}$$

The generalization of Mittag-Leffler function in two parameters α and β, we have

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z, \beta \in \mathbb{C} \quad and \quad Re(\alpha) > 0. \tag{2.12}$$

Definition 2.7. *[2] The k-Mittag-Leffler function with parameters $\alpha, \beta \in \mathbb{C}$ and $k \in \mathbb{R}$ is given as*

$$E^\eta_{k,\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!}, \qquad Re(\alpha) > 0, \, Re(\alpha) > 0 \quad z \in \mathbb{C}, \tag{2.13}$$

where $(\eta)_{n,k}$ is the Pochhammer symbol defined by

$$(\eta)_{n,k} = \eta(\eta + k)(\eta + 2k) \cdots (\eta + (n-1)k); \qquad n = 1, 2, \cdots. \tag{2.14}$$

3 Method of Solution

We shall introduce the so called the k-fractional Hilfer derivative as given in [19, 1], FRFT as given in [22] and an integral operator $\varepsilon^{\eta}_{k,\sigma,\lambda}\Phi$. Some of their properties will be stated in form of lemmas.

Definition 3.1. *Let f be sufficiently well-behaved function which have its support in $\mathbb{R}^+$ and let $\alpha > 0$ be a real number. The k-Riemann-Liouville fractional integral is given as*

$$I_k^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \tag{3.1}$$

where $\Gamma_k(\alpha)$ denotes the k-Gamma function given in (2.7).

Equivalently, we can write the above k-Riemann-Liouville fractional integral as a convolution

$$I_k^{\alpha} f(x) = \frac{1}{k\Gamma_k(\alpha)} [g(x) * f(x)], \tag{3.2}$$

where $g(x) = x^{\frac{\alpha}{k}-1}$ depending on both k and α.

Definition 3.2. *Let f be sufficiently well-behaved function which have its support in $\mathbb{R}^+$ and let $\alpha > 0$ be a real number. The k-Riemann-Liouville fractional derivative is given as*

$$D_k^{\alpha} f(x) = D I_k^{1-\alpha} f(x). \tag{3.3}$$

where operator $D = \frac{d}{dt}$.

Lemma 3.1. *Let f be sufficiently well-behaved function and let α be a real number, $0 < \alpha \leqslant 1$. The Laplace transform of the k-Riemann-Liouville fractional integral of the function f is given by*

$$\mathcal{L}\left\{I_k^{\alpha} f(x); s\right\} = (ks)^{-\frac{\alpha}{k}} \mathcal{L}\left\{f(x); s\right\}. \tag{3.4}$$

Proof. Using the properties of Laplace transform of convolution, we have that

$$\begin{aligned}
\mathcal{L}\left\{I_k^{\alpha} f(x); s\right\} &= \frac{1}{k\Gamma_k(\alpha)} \mathcal{L}\{g(x); s\} \mathcal{L}\{f(x); s\} \\
&= \frac{1}{k\Gamma_k(\alpha)} \left[\int_0^{\infty} x^{\frac{\alpha}{k}-1} e^{-st} dx\right] \mathcal{L}\{f(x); s\} \\
&= \frac{\Gamma\left(\frac{\alpha}{k}\right)}{k s^{\frac{\alpha}{k}} \Gamma_k(\alpha)} \mathcal{L}\{f(x); s\} \qquad (using\, lemma\, (2.3)) \\
&= \mathcal{L}\left\{I_k^{\alpha} f(x); s\right\} = (ks)^{-\frac{\alpha}{k}} \mathcal{L}\left\{f(x); s\right\} \qquad (using\, lemma\, (2.4)).
\end{aligned} \tag{3.5}$$

Therefore the result follows. □

Lemma 3.2. *Let f be sufficiently well-behaved function and let α be a real number, $0 < \alpha \leqslant 1$. The Laplace transform of the k-Riemann-Liouville fractional derivative of the function f is given by*

$$\mathcal{L}\left\{D_k^{\alpha} f(x); s\right\} = s^{\frac{\alpha+k-1}{k}} k^{\frac{\alpha-1}{k}} \mathcal{L}\left\{f(x); s\right\} - I_k^{1-\alpha} f(0^+). \tag{3.6}$$

Proof. The proof of this lemma is straight-forward by using the Laplace transform of classical derivative and the result obtained in Lemma (3.1), given below

$$\begin{aligned}
\mathcal{L}\left\{D_k^{\alpha} f(x); s\right\} &= \mathcal{L}\left\{D I_k^{1-\alpha} f(x); s\right\} \\
&= s\mathcal{L}\left\{I_k^{1-\alpha} f(x); s\right\} - I_k^{1-\alpha} f(0^+) \\
&= \frac{s\mathcal{L}\left\{f(x); s\right\}}{(ks)^{\frac{1-\alpha}{k}}} - I_k^{1-\alpha} f(0^+) \\
&= s^{\frac{\alpha+k-1}{k}} k^{\frac{\alpha-1}{k}} \mathcal{L}\left\{f(x); s\right\} - I_k^{1-\alpha} f(0^+).
\end{aligned} \tag{3.7}$$

Therefore the result follows. □

Definition 3.3. *Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$. We define*

$$ {}^{k}D^{\alpha,\beta} f(t) = \left(I_k^{(1-\alpha)\beta} \frac{d}{dt} \left(I_k^{(1-\alpha)(1-\beta)} f \right) \right) (t), \tag{3.8}$$

where

$$ I_k^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau) d\tau. \tag{3.9}$$

called k-Hilfer fractional derivative of order α and type β.
When $\beta = 0$ and $k = 1$, we obtain the Riemann-Liouville fractional derivative and for $\beta = k = 1$, we get Caputo fractional derivative.

Lemma 3.3. *[19] Using definitions (3.3), we have the followings*

$$ {}^{k}D^{\alpha,\beta} \left[(t-a)^{\frac{\rho}{k}-1} \right] (x) = \frac{\Gamma_k(\rho)}{k\Gamma_k(1-\alpha+\rho-k)} (x-a)^{\frac{1-\alpha+\rho-k}{k}-1} \tag{3.10}$$

$$ {}^{k}D^{\alpha,\beta} \left[(t-a)^{\frac{\rho}{k}-1} E_{k,\varrho,\rho}^{\sigma}(w(t-a)^{\frac{\varrho}{k}}) \right] (x) = \frac{(x-a)^{\frac{1-\alpha+\rho-k}{k}-1}}{k} E_{k,\varrho,\rho+1-\alpha-k}^{\sigma}(w(t-a)^{\frac{\varrho}{k}}) \tag{3.11}$$

Lemma 3.4. *[19]*

$$ \mathcal{L}\left\{ {}^{k}D^{\alpha,\beta} f(t); s \right\} = \frac{s\mathcal{L}\{f(t);s\}}{(ks)^{\frac{1-\alpha}{k}}} - \frac{I_k^{(1-\alpha)(1-\beta)} f(0^+)}{(ks)^{\frac{(1-\alpha)\beta}{k}}}, \tag{3.12}$$

for $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$.

Definition 3.4. *Let $\sigma, \lambda, \eta, \upsilon \in \mathbb{C}$, $Re\left(\frac{\sigma n+\lambda}{k}\right) > 1$. Define*

$$ \varepsilon_{k,\sigma,\lambda}^{\eta} \Phi(t) = \int_0^t (t-\tau)^{\frac{\lambda}{k}-1} E_{k,\rho,\lambda}^{\eta}(\upsilon(t-\tau)^{\frac{\rho}{k}}) \Phi(\tau) d\tau. \tag{3.13}$$

Lemma 3.5. *[19] The operator $\varepsilon_{k,\sigma,\lambda}^{\eta}\Phi$ defined above in (3.13) is bounded on the space of Lebesgue integrable functions, $L(a,b)$.*

Lemma 3.6. *[19] Let $\sigma, \lambda, \eta, \upsilon \in \mathbb{C}$, $Re(\sigma) > 0$, $Re(\lambda) > 0$, $Re(\eta) > 0$, $Re(s) > 0$, and $|k^{1-\frac{\sigma}{k}} \upsilon s^{\frac{-\rho}{k}}| < 1$. Then we have*

$$ \mathcal{L}\left\{ \varepsilon_{k,\sigma,\lambda}^{\eta}\Phi(t); s \right\} = \frac{s^{\frac{\sigma\eta}{k^2}} k^{1-\frac{\lambda}{k}}}{s^{\frac{\lambda}{k}} (s^{\frac{\sigma}{k}} - k^{1-\frac{\sigma}{k}}\upsilon)^{\frac{\eta}{k}}} \mathcal{L}\{\Phi(\tau); s\}. \tag{3.14}$$

Definition 3.5. *Let function f be of the class of a rapidly decreasing test functions on $\mathbb{R}$. The Fourier transform of f is defined as*

$$ \mathcal{F}\{f(x); p\} := f^*(p) = \int_{-\infty}^{+\infty} e^{i\alpha x} f(x) dx, \qquad p \in \mathbb{R}, \tag{3.15}$$

and the inverse Fourier transform is given as

$$ \mathcal{F}^{-1}\{f^*(p); x\} := f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ipx} f^*(p) dp, \qquad x \in \mathbb{R}. \tag{3.16}$$

Definition 3.6. *We define FRFT of a function, f_α^*, of order $0 < \alpha \leqslant 1$ as*

$$ \mathcal{F}_\alpha\{f(x); \alpha\} := f_\alpha^*(p) = \int_{-\infty}^{+\infty} e_\alpha(p,x) f(x) dx, \qquad p \in \mathbb{R}, \tag{3.17}$$

where

$$ e_\alpha(p,x) = \begin{cases} e^{-i|p|^{\frac{1}{\alpha}} x}, & p \leqslant 0 \\ e^{i|p|^{\frac{1}{\alpha}} x}, & p > 0. \end{cases} \tag{3.18}$$

Remark 2. *We can obtain the relationship between the FRFT and the classical Fourier transform as*

$$f_\alpha^*(p) = \mathcal{F}_\alpha \{f(x); p\} = \mathcal{F}_1 \{f(x); \omega\} = f^*(\omega), \tag{3.19}$$

where

$$\omega = \begin{cases} -i|p|^{\frac{1}{\alpha}}, & p \leqslant 0 \\ i|p|^{\frac{1}{\alpha}}, & p > 0. \end{cases} \tag{3.20}$$

So if

$$\mathcal{F}_\alpha \{f(x); p\} = \mathcal{F}_1 \{f(x); w\} = \phi(w), \tag{3.21}$$

then

$$f(x) = \mathcal{F}_\alpha^{-1} \{f_\alpha^*(p); x\} = \mathcal{F}_1^{-1} \{\phi(w); x\}. \tag{3.22}$$

Lemma 3.7. *Let $0 \leqslant \alpha < 1$ and $f^{(n)}(x) \in \phi(\mathbb{R})$. Then*

$$\mathcal{F}_\alpha \left\{f^{(n)}(x); p\right\} = (-isign(p)|p|^{\frac{1}{\alpha}})^n f_\alpha^*(p), \qquad p \in \mathbb{R}. \tag{3.23}$$

Lemma 3.8. *[22] Let $0 \leqslant \alpha \leqslant 1$, any value of β and a function $f(x) \in \phi(\mathbb{R})$. Then*

$$\mathcal{F}_\alpha \left\{D_\beta^\alpha f(x); p\right\} = (-iC_\alpha p)\mathcal{F}_\alpha \{f(x); p)\}, \qquad p \in \mathbb{R}, \tag{3.24}$$

where

$$C_\alpha = sin\left(\frac{\alpha\pi}{2}\right) + isign(p)(1-2\beta)cos\left(\frac{\alpha\pi}{2}\right). \tag{3.25}$$

4 Main Result

In this section, we present an application of FRFT and Laplace transform to solving Cauchy-type problems, involving k-fractional Hilfer derivative and an integral operator given in (3.13) that contain in its kernel the k-Mittag-Leffler function.

Theorem 4.1. *Let $f \in \phi(\mathbb{R})$, $0 < \alpha < 1$, $0 < \beta \leqslant 1$ and $0 < \gamma \leqslant 1$. The Cauchy type problem*

$$\begin{cases} {}^kD_t^{\alpha,\beta}u(x,t) = \delta D_\rho^{\gamma+1}\varepsilon_{k,\sigma,\lambda}^\eta g(x,t), \\ I_k^{(1-\alpha)(1-\beta)}u(x,0^+) = f(x). \end{cases} \tag{4.1}$$

where δ is constant, ${}^kD_t^{\alpha,\beta}$ is the k-fractional Hilfer derivative and $D_\rho^{\gamma+1}$ is the space fractional derivative in (3.24)*, is solvable for all values of $\rho \in \mathbb{R}$ and its solution $u(x,t)$ is given by*

$$u(x,t) = kI_k^{(1-\alpha)(1-\beta)+k}f(x) - i\delta kC_{\gamma+1}pt^{\frac{\lambda}{k}-1}E_{k,\sigma,\lambda}^\eta\left(\upsilon t^{\frac{\sigma}{k}}\right) * I_k^{\alpha+k-1}g(x,t). \tag{4.2}$$

Proof. Using lemma (3.8), we apply the FRFT, $f_{\gamma+1}^*$, in x to the equations in (4.1) and obtain

$${}^kD_t^{\alpha,\beta}u_{\gamma+1}^*(p,t) = -i\delta C_{\gamma+1}p\left(\varepsilon_{k,\sigma,\lambda}^\eta g_{\gamma+1}^*\right)(p,t), \tag{4.3}$$

and the corresponding initial condition in FRFT transform gives

$$I_k^{(1-\alpha)(1-\beta)}u_{\gamma+1}^*(p,0^+) = f_{\gamma+1}^*(p). \tag{4.4}$$

We can proceed to apply Laplace transform in t to equations (4.3) and (4.4), having in mind the results of lemma (3.4) and lemma (3.6), to get

$$\frac{s\mathcal{L}\{u_{\gamma+1}^*(p,t);s\}}{(ks)^{\frac{1-\alpha}{k}}} - \frac{f_{\gamma+1}^*(p)}{(ks)^{\frac{(1-\alpha)\beta}{k}}} = -\frac{i\delta C_{\gamma+1}ps^{\frac{\sigma\eta}{k^2}}k^{1-\frac{\lambda}{k}}}{s^{\frac{\lambda}{k}}(s^{\frac{\sigma}{k}} - k^{1-\frac{\sigma}{k}}\upsilon)^{\frac{\eta}{k}}}\mathcal{L}\{g_{\gamma+1}^*(p,t);s\}. \tag{4.5}$$

This gives

$$\frac{s\mathcal{L}\{u^*_{\gamma+1}(p,t);s\}}{(ks)^{\frac{1-\alpha}{k}}}=\frac{f^*_{\gamma+1}(p)}{(ks)^{\frac{(1-\alpha)\beta}{k}}}-\frac{i\delta C_{\gamma+1}ps^{\frac{\sigma\eta}{k^2}}k^{1-\frac{\lambda}{k}}}{s^{\frac{\lambda}{k}}(s^{\frac{\sigma}{k}}-k^{1-\frac{\sigma}{k}}\upsilon)^{\frac{\eta}{k}}}\mathcal{L}\{g^*_{\gamma+1}(p,t);s\}. \tag{4.6}$$

Hence,

$$\mathcal{L}\{u^*_{\gamma+1}(p,t);s\}=\frac{f^*_{\gamma+1}(p)(ks)^{\frac{1-\alpha}{k}}}{s(ks)^{\frac{(1-\alpha)\beta}{k}}}-\frac{i\delta C_{\gamma+1}ps^{\frac{\sigma\eta}{k^2}}k^{1-\frac{\lambda}{k}}(ks)^{\frac{1-\alpha}{k}}}{ss^{\frac{\lambda}{k}}(s^{\frac{\sigma}{k}}-k^{1-\frac{\sigma}{k}}\upsilon)^{\frac{\eta}{k}}}\mathcal{L}\{g^*_{\gamma+1}(p,t);s\}. \tag{4.7}$$

Making some re-arrangement, we get

$$\begin{aligned}\mathcal{L}\{u^*_{\gamma+1}(p,t);s\} &= \frac{kf^*_{\gamma+1}(p)}{(ks)(ks)^{\frac{(1-\alpha)\beta}{k}}(ks)^{\frac{\alpha-1}{k}}}-\frac{i\delta kC_{\gamma+1}ps^{\frac{\sigma\eta}{k^2}}k^{1-\frac{\lambda}{k}}}{(ks)s^{\frac{\lambda}{k}}(s^{\frac{\sigma}{k}}-k^{1-\frac{\sigma}{k}}\upsilon)^{\frac{\eta}{k}}(ks)^{\frac{\alpha-1}{k}}}\mathcal{L}\{g^*_{\gamma+1}(p,t);s\}\\ &= \frac{kf^*_{\gamma+1}(p)}{(ks)^{\frac{(1-\alpha)(1-\beta)+k}{k}}}-\frac{i\delta kC_{\gamma+1}ps^{\frac{\sigma\eta}{k^2}}k^{1-\frac{\lambda}{k}}}{s^{\frac{\lambda}{k}}(s^{\frac{\sigma}{k}}-k^{1-\frac{\sigma}{k}}\upsilon)^{\frac{\eta}{k}}}\frac{\mathcal{L}\{g^*_{\gamma+1}(p,t);s\}}{(ks)^{\frac{\alpha+k-1}{k}}}. \end{aligned} \tag{4.8}$$

Using lemma (3.1), we obtain

$$\begin{aligned}\mathcal{L}\{u^*_{\gamma+1}(p,t);s\} &= k\mathcal{L}\left\{I_k^{(1-\alpha)(1-\beta)+k}f^*_{\gamma+1}(p);s\right\}\\ &\quad -i\delta kC_{\gamma+1}p\mathcal{L}\left\{t^{\frac{\lambda}{k}-1}E^{\eta}_{k,\sigma,\lambda}\left(\upsilon t^{\frac{\sigma}{k}}\right);s\right\}\mathcal{L}\left\{I_k^{\alpha+k-1}g^*_{\gamma+1}(p,t);s\right\}. \end{aligned} \tag{4.9}$$

Taking the inverse Laplace transform of (4.9) and using the property of Laplace transform of convolution, we have

$$u^*_{\gamma+1}(p,t)=kI_k^{(1-\alpha)(1-\beta)+k}f^*_{\gamma+1}(p)-i\delta kC_{\gamma+1}p\left[t^{\frac{\lambda}{k}-1}E^{\eta}_{k,\sigma,\lambda}\left(\upsilon t^{\frac{\sigma}{k}}\right)*I_k^{\alpha+k-1}g^*_{\gamma+1}(p,t)\right].$$

Due to the fact from (3.19) and (3.20), the above latest quantity becomes

$$u^*(\omega,t)=kI_k^{(1-\alpha)(1-\beta)+k}f^*(\omega)-i\delta kC_{\gamma+1}p\left[t^{\frac{\lambda}{k}-1}E^{\eta}_{k,\sigma,\lambda}\left(\upsilon t^{\frac{\sigma}{k}}\right)*I_k^{\alpha+k-1}g^*(\omega,t)\right].$$

Then we apply inverse Fourier transform and observing that the first component of the convolution in the above latest quantity does not depend on ω to obtain

$$u(x,t)=kI_k^{(1-\alpha)(1-\beta)+k}f(x)-i\delta kC_{\gamma+1}pt^{\frac{\lambda}{k}-1}E^{\eta}_{k,\sigma,\lambda}\left(\upsilon t^{\frac{\sigma}{k}}\right)*I_k^{\alpha+k-1}g(x,t).$$

The above can be computed using (3.1) and then we have the desired solution. □

Theorem 4.2. *Let f be sufficiently well-behaved function which have its support in $\mathbb{R}^+$, $0<\alpha<1$ and $0<\beta\leqslant 1$. The Cauchy type problem*

$$\begin{cases} {}^kD_t^{\alpha,\beta}u(x)=\mu D_k^{\alpha}u(x)+f(x),\\ I_k^{(1-\alpha)(1-\beta)}u(0^+)=C_1 \qquad and \qquad I_k^{(1-\alpha)}u(0^+)=C_2\end{cases} \tag{4.10}$$

where ${}^kD_t^{\alpha,\beta}$ is the k-fractional Hilfer derivative and D_k^{α} is k-Riemann-Liouville fractional derivative in (3.3)*, is solvable for all values of $\mu\in\mathbb{R}-\{0,1\}$ and its solution $u(x)$ is given by*

$$u(x)=\frac{k}{1-\mu}\left[I_k^{(1-\alpha)(1-\beta)+k}C_1-\mu I_k^{\alpha+k-1}C_2+I_k^{\alpha+k-1}f(x)\right]. \tag{4.11}$$

Proof. We apply Laplace transform to equations (4.10), using the results of lemma (3.2) and lemma (3.4), to get

$$\frac{s\mathcal{L}\{u(x);s\}}{(ks)^{\frac{1-\alpha}{k}}} - \frac{C_1}{(ks)^{\frac{(1-\alpha)\beta}{k}}} = \mu s(ks)^{\frac{\alpha-1}{k}}\mathcal{L}\{u(x);s\} - \mu C_2 + \mathcal{L}\{f(x);s\}. \tag{4.12}$$

This gives

$$\frac{s-\mu s}{(ks)^{\frac{1-\alpha}{k}}}\mathcal{L}\{u(x);s\} = \frac{C_1}{(ks)^{\frac{(1-\alpha)\beta}{k}}} - \mu C_2 + \mathcal{L}\{f(x);s\}. \tag{4.13}$$

Hence,

$$\mathcal{L}\{u(x);s\} = \frac{kC_1(ks)^{\frac{1-\alpha}{k}}}{(1-\mu)(ks)(ks)^{\frac{(1-\alpha)\beta}{k}}} - \frac{\mu kC_2(ks)^{\frac{1-\alpha}{k}}}{(ks)(1-\mu)} + \frac{k(ks)^{\frac{1-\alpha}{k}}}{(ks)(1-\mu)}\mathcal{L}\{f(x);s\}. \tag{4.14}$$

Making some re-arrangement, we get

$$\mathcal{L}\{u(x);s\} = \frac{k}{1-\mu}\left[\frac{C_1}{(ks)^{\frac{(1-\alpha)(1-\beta)+k}{k}}} - \frac{\mu C_2}{(ks)^{\frac{\alpha+k-1}{k}}} + \frac{1}{(ks)^{\frac{\alpha+k-1}{k}}}\mathcal{L}\{f(x);s\}\right]. \tag{4.15}$$

Applying lemma (3.1), we obtain

$$\mathcal{L}\{u(x);s\} = \frac{k}{1-\mu}\left[\mathcal{L}\left\{I_k^{(1-\alpha)(1-\beta)+k}C_1;s\right\} - \mu\mathcal{L}\left\{I_k^{\alpha+k-1}C_2;s\right\} + \mathcal{L}\left\{I_k^{\alpha+k-1}f(x);s\right\}\right]. \tag{4.16}$$

The inverse Laplace transform of (4.16) then gives

$$u(x) = \frac{k}{1-\mu}\left[I_k^{(1-\alpha)(1-\beta)+k}C_1 - \mu I_k^{\alpha+k-1}C_2 + I_k^{\alpha+k-1}f(x)\right]. \tag{4.17}$$

The above can be computed using (3.1) and then we have the desired solution. □

5 Conclusion

The solvability of Cauchy-type problems involving k-fractional Hilfer derivatives and an integral operator with a k-Mittag-Leffler function appearing in the Kernel is made possible using the combination of FRFT method and the classical Laplace transform method. The elegant nature of Mittag-Leffler function gives the form of solution presented an easy way to describe it. As a matter of fact, the combination of fractional Fourier transform method and Laplace transform method presents a wide applicability to handling related fractional Cauchy-type problems and some special type of fractional models containing a k-Mittag-Leffler function in their the Kernel.

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Competing Interests

Author has declared that no competing interests exist.

Authors' Contributions

All author contributed equally and significantly. All author read and approved the final manuscript.

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