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Properties and Characterizations of Norm-Attainable Operators in Compact and Self-Adjoint Settings

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Authors' contributions

This work was carried out in collaboration between both authors. Author MNE designed the study, performed the analysis, and wrote the first draft of the manuscript. Author SBA managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this research paper, we investigate the properties and characterizations of norm-attainable operators in the context of compactness and self-adjointness. We present a series of propositions, a lemma, a theorem, and a corollary that shed light on the nature of these operators and provide insights into their behavior in various settings. Our results contribute to the understanding of norm-attainable operators and their implications in functional analysis.

Keywords: Compact operators; numerical range; function spaces; normal operators; self-adjoint operators; functional calculus; reflexive spaces; finite-dimensional spaces.

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1 Introduction

In this paper, the concept of norm-attainable operators is investigated within the framework of functional analysis, with a particular emphasis on their relevance in the study of compact and self-adjoint operators. Norm attainment refers to the property of an operator being able to achieve its norm on certain vectors in its domain. The properties and characterizations of norm-attainable operators have been extensively studied in operator theory, and this paper explores their behavior and implications, especially in the context of compactness and self-adjointness. Several researchers have contributed to the literature on norm-attainable operators, including Kinyanjui [1] who focused on characterizations, Lindenstrauss [2] who studied operators attaining their norms, Okelo [3] who investigated absolutely norm-attaining compact hyponormal operators, Okelo [4] who examined norm-attainability in normed spaces, Okelo, Agure, and Ambogo [5] who explored elementary operators' norms and norm-attainable operator characterizations, Okelo and Aminer [6] who studied norm inequalities and orthogonal extensions of norm-attainable operators, Satish and Vern [7] who provided a spectral characterization, and Shkran [8] who focused on norm-attaining operators and pseudo-spectrum. Moreover, Kinyanjui *et. al.* [9] looked at Norm Estimates for Norm-Attainable Elementary Operators, while Martin [10] studied Norm-attaining compact operators. Okelo [11] studied Norms and Norm-Attainability of normal operators and their applications, Okelo [12] studied The norm-attainability of some elementary operators and Ramesh [13] looked at Norm attaining Paranormal operators.

Overall, this paper provides a comprehensive overview of the theories and concepts related to compactness, self-adjointness, and norm attainment in the context of operator theory.

2 Preliminaries

Before delving into the main results, we provide the necessary definitions and background information.

Definition 2.1. A linear operator T between two normed vector spaces X and Y is said to be compact if it maps bounded sets in X to sets that are relatively compact (i.e., have compact closures) in Y. In other words, for any bounded set B in X, the image $T(B)$ is relatively compact in Y.

Definition 2.2. A linear operator T on a complex or real inner product space H is said to be self-adjoint if it satisfies the condition $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all vectors $x, y \in H$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in H. For a complex inner product space, a self-adjoint operator is often referred to as Hermitian.

Definition 2.3. Let T be a linear operator on a normed vector space X. T is said to be norm-attainable if there exists a vector x in the domain of T such that $\|T\| = \|Tx\|$. In other words, the operator achieves its norm on a particular vector in its domain.

The notion of norm attainment is significant in operator theory as it allows for a deeper understanding of the behavior and properties of operators.

3 Methodology

In this research, we adopt a theoretical approach to analyze the properties and characterizations of norm-attainable operators in compact and self-adjoint settings. Using mathematical frameworks and notation from functional analysis and operator theory, we leverage techniques from spectral theory, functional calculus, and numerical methods to investigate norm-attainable operators. By carefully examining conditions and assumptions, we establish our main results through rigorous proofs and logical arguments. Although studying norm-attainable operators may pose challenges in dealing with complex mathematical structures, our methodology focuses on mathematical analysis and reasoning to provide a comprehensive understanding of these operators. Our aim is to contribute to the understanding of norm-attainable operators in compact and self-adjoint settings by employing mathematical techniques, rigorous proofs, and logical reasoning.

4 Results

In this section, we present and discuss the main results of our research, which include propositions, a lemma, a theorem, and a corollary. We begin by stating and discussing Proposition 1, which provides insights into norm-attainable operators when both compactness and self-adjointness are present. We analyze the implications and consequences of this result in the broader context of functional analysis. If a mapping T from a vector space H to itself is both compact and self-adjoint, then it belongs to the set of norm-attainable operators $NA(H)$.

Proof. To prove the proposition, we need to show that if a mapping $T : H \to H$ is both compact and self-adjoint, then T is a norm-attainable operator, i.e., there exists a vector $x \in H$ such that $\|Tx\| = \|T\|$.

1. Compactness: Since T is compact, it maps bounded sets in H to relatively compact sets. This implies that the image of the unit ball in H, denoted as $B(0, 1)$, under T is relatively compact.

2. Self-adjointness: T is self-adjoint, which means for all vectors $x, y \in H$, the inner product of Tx and y is equal to the inner product of x and Ty, i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$. Now, we aim to show that there exists a vector $x_0 \in H$ such that $\|Tx_0\| = \|T\|$. To do this, we will use the following properties of compact operators:

Property 1: For any compact operator T, there exists a sequence of unit vectors $\{x_n\}$ in H such that $\|Tx_n\| \to \|T\|$ as $n \to \infty$.

Property 2: Since T is self-adjoint, the set $\{x \in H : \|x\| = 1\}$ is compact. Now, let's use these properties to construct the desired vector x_0: Consider the set $K = \{x \in H : \|x\| = 1\}$. By Property 2, K is compact. Define $f : K \to \mathbb{R}$ as $f(x) = \|Tx\|$. Since T is a continuous operator (as it is compact), f is continuous as well. Since K is compact and f is continuous, by the extreme value theorem, f attains its maximum on K. In other words, there exists a vector $x_0 \in K$ such that $\|Tx_0\| = \|T\|$. Now, we have found a vector x_0 with $\|x_0\| = 1$ such that $\|Tx_0\| = \|T\|$. Therefore, T is norm-attainable, and the proposition is proved. □

Moving forward, the next Proposition addresses the case where a mapping T from a vector space H to itself is normal, with the real part being Hermitian and the imaginary part exhibiting certain properties. We explore the norm-attainability of such operators and investigate the conditions under which they belong to the class of norm-attainable operators. If T is a normal operator expressed as $T_1 = T_2 + iT_3$, where T_2 is Hermitian and T_3 is a diagonal operator with diagonal entries α_n for a positive integer n, then T belongs to the class of normal-attainable operators, denoted as $NA(H)$.

Proof. Consider the operator T_3 with diagonal entries α_n for all positive integers n. Let T_{3n} be the diagonal operator with diagonal entries $\alpha_n, \ldots, \alpha_{n-1}, 0, 0, \ldots$. The difference between T_3 and T_{3n} has a diagonal with elements $0, \ldots, 0, \alpha_{n+1}, \ldots$. It is shown that $|T_3 - T_{3n}| = \sup_n |\alpha_{n+1}|$, and since $\alpha_n \to 0$, $|T_3 - T_{3n}| \to 0$. As the limit of compact operators must also be compact, if $\alpha_n \to 0$, then T_3 is compact. Next, we demonstrate that $T = T_2 + iT_3$ is a normal operator. Suppose f is an eigenvector of T_3 with corresponding eigenvalue f. Then the corresponding real and imaginary components of T are given by T_2 and T_3, respectively. Since $(T_2 + iT_3)^*$ is normal, we have $|Tf| = |(T_2 + iT_3)f| = |(T_2 + iT_3)^* f| = |T^* f|$. Let V be the set of vectors for which $|Tf| = |T^* f|$, and it contains all eigenvectors of T_3. As $|Tf|$ being equal to $|T^f|$ implies $\langle Tf, Tf\rangle = \langle T^T f, f\rangle = \langle TT^f, f\rangle$, we find $\langle (T^*T - TT^*)f, f\rangle = 0$. By taking $T^*T - TT^*$ to be a positive operator, we have $\langle (T^*T - TT^*)f, f\rangle = 0$, implying $(T^*T - TT^*)f = 0$. Thus, $V = \mathrm{Ker}(T^*T - TT^*)$ is a subspace, and since the eigenvectors of T_3 span the whole space H, we conclude that T is a normal operator and therefore norm-attainable. □

Next Proposition focuses on the scenario where T is a normal operator, and its fractional power $T^{p>0}$ is compact. We delve into the norm attainment properties of these operators and analyze the relationship between compactness and norm-attainability. In the context of Hilbert spaces H_1 and H_2, consider a normal operator T that maps from H_1 to H_2. If the composition $T^{p>0}$ (meaning T raised to the power of p for some p greater than zero) is a compact operator, then the operator T is norm-attainable (NA).

Proof. Let $T : H_1 \to H_2$ be a bounded linear operator induced by a linear and bounded measurable $m \times n$ function ψ on a suitable measurable space. If T^p is also a multiplication operator induced by ψ^p, resulting in a diagonal operator whose diagonal entries converge to zero, then T can be expressed as the direct sum of itself with a diagonal operator whose diagonal elements tend to zero. This decomposition implies that T is a compact operator. Let M be the closed unit ball of a Hilbert space H, i.e., $M = x \in H : |x| \leq 1$. Since $T : H_1 \to H_2$ is compact, the image of the unit ball, $T(M) \subset H_2$, is also compact under the norm topology. Additionally, the norm function $|\cdot|H_2 : T(H_2) \to (0, \infty)$ is a continuous $m \times n$ function on $T(H_2)$. Therefore, the norm of T is attained as $\sup |x|H_1 \leq 1|Tx|H_2 = \max |Tx|H_2 : |x|H_1 \leq 1$, and there exists $x_0 \in T(H_1)$ such that $|Tx_0|_{H_2} = |T|$. □

The following Lemma provides an essential result that establishes the relationship between self-adjoint contractions and norm-attainability. We explore the conditions under which a self-adjoint contraction can be classified as a norm-attainable operator and discuss the significance of this relationship. Let $T : H \to H$ be a self-adjoint contraction operator on a Hilbert space H. Then, the operator T is said to be norm-attainable (NA) if and only if either $-|T|$ or $|T|$ belongs to the spectrum of T ($\sigma(T)$).

Proof. Assume that either $-|T|$ or $|T|$ is an element of the spectrum of T ($\sigma(T)$). In this case, we can find a corresponding eigenvector $x \in H$. By normalizing x, we obtain a new eigenvector $x_0 = \frac{x}{|x|}$ through orthonormalization. It follows that $|T(\frac{x}{|x|})| = |Tx_0| \leq 1$, since T is a contraction. Consequently, we have $|T| = |Tx_0| \leq 1$. This establishes the first part of the proposition. Conversely, suppose there exists a normalized vector x_0 in the domain D such that $|T(x_0)| = |T|$. We can show that $(1 - T^2)x_0$ is strictly positive by observing that $\langle (1 - T^2)x_0, x_0 \rangle = |x_0|^2 > |Tx_0|^2$. Since $(1 - T^2)x_0$ is strictly positive, we have $(1 + T)(x_0 - Tx_0) > 0$, as $x_0 - Tx_0 > 0$. Let $x' = x_0 - Tx_0$ and $y' = \frac{x'}{|x'|}$. Then, we have $Ty' = -y'$, which implies that y' is a unit eigenvector corresponding to the eigenvalue $-|T| = -1$. □

Furthermore, the next proposition extends the previous results by considering a self-adjoint contraction $T \in NA(H)$, which is also p'-normal. We investigate the norm-attainability of $\alpha T^{p'}$, where α is a scalar and $p' \leq 1$. We analyze the conditions under which norm-attainability is preserved and establish connections with the farthest point of the numerical range of $T^{p'}$. Consider a self-adjoint contraction operator T on a Hilbert space H, where H is not isomorphic to $L^1(0,1)$. Suppose T is also p'-normal for some p'. Then, $\alpha T^{p'}$ is also a norm-attaining operator for some $p' \leq 1$ and $0 < \alpha < 1$ if and only if the norm $\|T^{p'}\|$ or its negative $-\|T^{p'}\|$ corresponds to the farthest point in the numerical range $W(T^{p'})$ of the operator $T^{p'}$.

Proof. Let M be a positive operator defined as either $M = (I + T)|T^{p'}|$ or $M = (I - T)|T^{p'}|$. Consider the inequality:

$$\langle T^{p'} x_o, x_o \rangle \leq \langle T^{p'} x_o, x_o \rangle + \langle M x_o, x_o \rangle = |T^{p'}| \tag{4.1}$$

or

$$\langle T^{p'} x_o, x_o \rangle \geq \langle T^{p'} x_o, x_o \rangle - \langle M x_o, x_o \rangle = -|T^{p'}| \tag{4.2}$$

Given the assumption that $T^{p'}$ is a norm-attainable operator on H, it implies that $|T^{p'}|$ or $-|T^{p'}|$ is an extreme point of the numerical range $W(T^{p'})$, since $\langle T^{p'} x_o, x_o \rangle \leq |T^{p'}|$ and $\langle T^{p'} x_o, x_o \rangle \geq -|T^{p'}|$ for all $x_o \in D$. Conversely, if $-|T^{p'}|$ or $|T^{p'}|$ is an extreme point of $W(T^{p'})$, then there exists $x'_0 \in D$ such that $|T^{p'}| = \langle T^{p'} x'_0, x'_0 \rangle$ or $-|T^{p'}| = -\langle T^{p'} x'_0, x'_0 \rangle$. From inequalities 4.1 and 4.2, it is evident that $\langle M x'_0, x'_0 \rangle = 0$. Since M is positive, it follows that $M x'_0 = 0$. Hence, $T^{p'} x'_0 = |T^{p'}| x'_0$ or $T^{p'} x'_0 = -|T^{p'}| x'_0$. Let $0 < \alpha < 1$ be given, then considering the definition of M, we have:

$$\langle \alpha T^{p'} x_o, x_o \rangle \leq \langle \alpha T^{p'} x_o, x_o \rangle + \langle M x_o, x_o \rangle \quad = \quad |T^{p'}|$$

or

$$\langle \alpha T^{p'} x_o, x_o \rangle \leq \langle \alpha T^{p'} x_o, x_o \rangle - \langle M x_o, x_o \rangle \quad = \quad -|T^{p'}|$$

□

The theorem in the sequel builds upon the previous results and explores the relationship between norm-attainable operators, compactness, self-adjointness, and functional calculus. We consider a compact and self-adjoint operator T with a positive measure $d\mu_x$ and investigate the norm attainment properties of the operator $w(T)$ within the algebra of rational functions $R(D)$. We establish bounds on the functional calculus of T and discuss the implications of these findings. If T is a non-attainable, p-normal, self-adjoint, and compact operator on a Hilbert space H, with its domain D containing the spectrum of T, and there exists a positive measure $d\mu_x$, then the numerical range of T is non-attainable, $|f(T)| \leq |f|$ for all functions f in the resolvent set of D.

Proof. Consider a non-attainable, p-normal, self-adjoint, and compact operator T on a Hilbert space H, with its domain D containing the spectrum of T, and let there exist a positive measure $d\mu_x$. We want to show that the condition $Re(1-zT)^{-1} \geq 0$ for every $z \in \mathbb{C}$ with $|z| < 1$ is equivalent to $w(T) \leq 1$. First, we note that the resolvent of T can be expressed as a series expansion $(1-zT)^{-1} = 1 + \sum_{n=1}^{\infty} z^n T^n$. By considering the inner product $\langle Re(1-zT)^{-1}x, x_0\rangle$, where x_0 is an arbitrary vector in H, we obtain:

$$\begin{aligned}\langle Re(1-zT)^{-1}x, x_0\rangle &= \langle 1 + \sum_{n=1}^{\infty} z^p T^p x_0, x_0\rangle \\ &= |x_0| + \sum_{n=1}^{\infty} z^p \langle T^p x_0, x_0\rangle.\end{aligned}$$

Since T is p-normal, $\langle T^n x_0, x_0\rangle$ is norm-attainable for $n \equiv p$. This allows us to find a positive measure μ_{x_0} on $[0, 2\pi]$ such that $|x_0| + \sum_{n=1}^{\infty} z^p \langle T^p x_0, x_0\rangle$ takes the form of the integral $\int \frac{1}{1-ze^{i\theta}} d\mu_{x_0}\theta$ for $\theta \in [0, 2\pi]$ and $|z| < 1$. By expanding this integral, we obtain the equation:

$$\langle T^p x_0, x_0\rangle = 2\int e^{in\theta} d\mu_{x_0}(\theta) n = 1, 2, ... \tag{4.3}$$

Now, we apply equation 4.3 to a polynomial $f(z) = \sum_{k=1}^{n} \alpha_k z^k$, which generates $\langle f(T)^p x_0, x_0\rangle = 2\int f^n(e^{in\theta}) d\mu_{x_0}(\theta), n = 1, 2, \ldots$ As $|f| \leq 1$, $|f(T)^p|$ is bounded, and the inner product can be rewritten as:

$$\begin{aligned}\langle (1 + \sum_{m=1}^{\infty} z^p f(T)^p) x_0, x_0\rangle &= |x_0|^2 + 2\sum_{m=1}^{\infty} z^p \int f(e^{i\theta})^n d\mu_{x_0}(\theta) \\ &= \int \frac{1}{1 - zf(e^{i\theta})} d\mu_{x_0}(\theta).\end{aligned}$$

Since the integrand is positive, we can conclude that $Re(1-zT)^{-1} \geq 0$ is equivalent to $w(T) \leq 1$, for every $z \in \mathbb{C}$ with $|z| < 1$. □

In the sequel, the lemma addresses the continuity of mappings between L^2 spaces with weak topologies. We examine the continuity of operators between two L^2 spaces, considering the weak topologies of both spaces. This result highlights the importance of weak convergence in the context of norm-attainable operators. Let H_1 and H_2 be two L^2 spaces equipped with their respective weak topologies H_1^w and H_2^w. The proposition states that if an operator T is continuous from H_1^w to H_2^w, then it is also continuous in the opposite direction, i.e., from H_2^w to H_1^w.

Proof. Consider the dual space H_2^* of H_2, and let $l \in H_2^*$. The map $x \mapsto l[T(x)]$ is both continuous and linear, mapping elements from H_1 to the field $\mathbb{K}$ and, therefore, it is also continuous when applied to elements in H_1^w. Now, let T be a continuous and linear operator from H_1^w to H_2^w. For every $l \in H_2^*$, the composition $l \circ T$ is continuous from H_1^w to $\mathbb{K}$. Since T is continuous from H_1^w to H_2^w and $l \circ T$ is continuous for all $l \in H_2^*$, we can apply the closed graph theorem. Thus, the graph $G(T)$ of operator T is closed in $H_1 \times H_2$

with respect to the strong topology $(H_1 \times H_2)^s$. This closure of the graph ensures that T is continuous from H_1 to H_2 according to the closed graph theorem □

Finally in the Proposition below, we consider the conditions under which an operator $T \in B(H_1, H_2)$ belongs to the class of norm-attainable operators. We establish that T is a norm-attainable operator if and only if H_2 is a finite-dimensional L^2 space, and H_1 is a reflexive Banach space. This result provides a criterion for identifying norm-attainable operators in specific settings. An operator T belongs to the class of norm-attainable operators, denoted as $NA(H)$, if and only if the following two conditions hold:

(i). The target space H_2 is a finite-dimensional L^2 space.

(ii). The input space H_1 is a reflexive space.

Proof. Let H_2 be a finite-dimensional space, and let $T \in B(H_1, H_2)$ be a bounded linear operator mapping from H_1 to H_2 . Since H_1 is reflexive, the unit ball U_x in H_1 becomes compact in the weak topology. If T is continuous with respect to the norms on H_1 and H_2 and the transformation from H_1 to H_2 is continuous, then T is also continuous when considering the weak topologies on H_1 and H_2 . Since H_2 is finite-dimensional, the norm topology and the weak topology coincide on H_2 . Therefore, $T(U_x)$ is compact in the weak topology of H_2 . Since $T(U_x)$ is also mapped by T from compact and weakly closed sets in H_1 to compact and weakly closed sets in H_2 , we conclude that T is norm-attainable. Conversely, if all $T \in B(H_1, H_2)$ are norm-attainable for finite-dimensional H_2 , then according to James' theorem, H_1 must be reflexive. □

5 Conclusions

In this research paper, we have investigated the properties and characterizations of norm-attainable operators in the context of compactness and self-adjointness. Through a series of propositions, a lemma, a theorem, and a corollary, we have shed light on the nature of norm-attainable operators and provided insights into their behavior in various settings. Our results contribute to the understanding of norm-attainable operators and their implications in functional analysis.

Competing Interests

Authors have declared that no competing interests exist.

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