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External Bifurcations of Double Heterodimensional Cycles with One Orbit Flip

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Abstract

In this paper, external bifurcations of heterodimensional cycles connecting three saddle points with one orbit flip, in the shape of "∞", are studied in three-dimensional vector field. We construct a poincaré return map between returning points in a transverse section by establishing a locally active coordinate system in the tubular neighborhood of unperturbed double heterodimensional cycles, through which the bifurcation equations are obtained under different conditions. Near the double heterodimensional cycles, the authors prove the preservation of "∞"-shape double heterodimensional cycles and the existence of the second and third shape heterodimensional cycle and a large 1-heteroclinic cycle connecting with p_1 and p_3. The coexistence of a 1-fold large 1-heteroclinic cycle and the "∞"-shape double heterodimensional cycles and the coexistence conditions are also given in the parameter space.

Keywords

Double Heteroclinic Loops, Orbit Flip, Heteroclinic Bifurcation, Bifurcation Theory

1. Introduction

In recent years, bifurcation theory has been widely concerned due to its importance in practical applications (see [1] [2] [3] [4]) and in the study of traveling wave solutions for nonlinear partial differential equations. For example, in 2018, Zilburg and Rosenau [5] studied the qualitative properties of solitons of a dKdV equation,

$$\partial_t u + \partial_x \left(u \partial_x u \right) + u^2 = 0. \tag{1.1}$$

Then Zhang [6] analyzed (1.1) in the idea of bifurcation theory of dynamical system. Roughly to speak, he first set the variable transform $u(x,t)=\phi(x-ct)=\phi(\xi)$ to make system (1.1) be

$$-c\phi'+\left(\phi(\phi\phi')'+\phi^2\right)'=0, \tag{1.2}$$

then integrated (1.2) and got

$$-c\phi+\phi\phi'^2+\phi^2\phi''+\phi^2=g, \tag{1.3}$$

where g is the integral constant, and system (1.3) is equivalent to the following regular plane system with $\mathrm{d}\xi=\phi^2\mathrm{d}\tau$

$$\begin{cases}\dfrac{\mathrm{d}\phi}{\mathrm{d}\tau}=y\phi^2\\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau}=g+c\phi-\phi y^2-\phi^2.\end{cases} \tag{1.4}$$

Clearly the Hamiltonian of system (1.4) is

$$H(\phi,y)=\frac{1}{2}y^2\phi^2-g\phi-\frac{1}{2}c\phi^2+\frac{1}{3}\phi^3=h. \tag{1.5}$$

From $H(\phi,y)\big|_{S1}=h_1$, a heteroclinic orbit is found as

$$y^2=\frac{2h_1+2g\phi+c\phi^2-\frac{2}{3}\phi^3}{\phi^2}$$

for $g=0, c>0$, and the existing condition is given in some circumstances on two sides of the nonresonant heteroclinic bifurcation.

In fact, different kinds of high co-dimensional homoclinic or heteroclinic bifurcations have been discussed extensively. [7] described a phenomenon that occurred in the bifurcation theory of one-parameter families of diffeomorphisms. If all the equilibrium points of the orbit have the same dimension number of the stable manifold, the heteroclinic cycle is named as an equidimensional loop, otherwise, a heterodimensional. However, since different equilibrium points in n-dimensional systems do not necessarily have stable manifolds of the same dimension, the problem of heterodimensional loop is more general and practical than that of equidimensionals. Jens D.M in [8] considered a self-organized periodic replication process of travelling pulses which has been observed in reaction-Cdiffusion equations, and studied homoclinic orbits near codimension-1 and -2 heteroclinic cycles between an equilibrium and a periodic orbit for ordinary differential equations in three or higher dimensions. Bykov analyzed the bifurcations of systems close to systems having contours composed of separatrices of a pair of saddle points (see [9]). [10] studied the bifurcations of heterodimensional cycles with the connection of two hyperbolic saddle points and strong inclination flip in a four-dimensional system, they presented the conditions for the existence, coexistence and noncoexistence of the heterodimensional orbit, homoclinic orbit and periodic orbit, as well as the co-existence of heterodimensional orbit and homoclinic orbit and obtained some new features from the inclination flip in some bifurcation surfaces. Xu and Lu discussed heterodimensional loop bifurcation with orbit flip and inclination flip respectively in [11] [12] [13], and got the coexistence region of coexisting loop and periodic orbit. Meanwhile, they also constructed an example to provide a good reference for their main bifurcation problems. Specially, Liu's team fabricated a model of heterodimensional cycles to verify their main bifurcation results (see [14] [15] [16] [17]).

However in the study of systems with homoclinic loop or heteroclinic loop, few scholars focused on double heteroclinic bifurcation of three saddle points. We only found that [18] considered the bifurcation problem of rough heteroclinic loops connecting three saddle points, but not a "∞"-type, for a higher-dimensional system and [19] concerned "∞"-type double homoclinic loops, but not heteroclinic loops, with resonance characteristic roots in the common case and in a four-dimensional system to obtain the complete bifurcation diagram under different conditions. In this paper, we consider the bifurcation problem of double heteroclinic loops of ∞-type connecting three saddle points with four orbits. In addition, we also give an example model to demonstrate the existence of the bifurcation results.

It's worth noting that, in the previous studies about homoclinic and heteroclinic loop bifurcations, few scholars focused on double heterodimensional cycles bifurcations of three saddle points. Jin and Zhu [18] considered the bifurcation problem of rough heteroclinic loop connecting three saddle points in a higher-dimensional system, but the loop is not a "∞"-type. [20] [21] [22] [23] discussed the heteroclinic loops with two saddle points, but the loops are not heterodimensional cycles. Lu and Liu *et al.* [10] [11] [13] studied the heterodimensional cycle, but the cycle is also neither a "∞"-type nor double. Jin *et al.* [19] [24] considered "∞"-type double homoclinic loops, but the loops are not heteroclinic or do not connect with three saddle points. Since heterodimensional or heteroclinic cycles are very normal and have applications in solitary wave problems and biology systems, see Kalyan Manna *et al.* [25] for example, and also for the completeness of theoretical research of heteroclinic bifurcation, in this paper, we focus on the double heterodimensional cycles in ∞-type with three saddle points.

The rest of the paper is structured as follows. In Section 2, through establishing a local moving frame system near the unperturbed heterodimensional cycle to obtain the Poincaré map and the successor function, we induce the bifurcation equations by using the implicit function theorem. Section 3 will show the bifurcation results on different parameter regions by analyzing the bifurcation equation.

The C^r system to be studied is

$$\dot{Z} = f(z) + g(z,\mu), \tag{1.6}$$

where $z \in R^3, \mu \in R^l, l > 4, |\mu| \ll 1, f, g \in C^r, r \geq 4$. Specially, when $\mu = 0$, the unperturbed system associated with (1.6) is

$$\dot{Z} = f(z). \tag{1.7}$$

satisfies the following hypotheses.

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(H₁) (*Hyperbolic*) $z = p_i\left(i = 1,2,3\right)$ are hyperbolic critical points of (1.7) such that $g\left(p_i,\mu\right) = g\left(p_i,\mathbf{0}\right) = \mathbf{0}$ for all *i*, and $\dim\left(W_1^s\right) = \dim\left(W_2^u\right) = \dim\left(W_3^u\right) = 2$ where $\mathbf{0}$ means a zero vector. In addition, the linearization matrix $D_z f\left(p_i\right)$ has a simple real eigenvalues: $-\rho_i^2, -\rho_i^1, \lambda_i^1\left(i = 1,3\right), -\rho_2^1, \lambda_2^1, \lambda_2^2$ satisfying

$$-\rho_i^2 < -\rho_i^1 < 0 < \lambda_i^1, -\rho_2^1 < 0 < \lambda_2^1 < \lambda_2^2$$

Throughout the paper we assume that system (1.7) is of at least C^3 uniformly linearizable. What's more, there is a small neighborhood $U_i\left(i = 1,2,3\right)$ of the equilibrium p_i and a C^3 diffeomorphism depending on the parameter in C^3 manner, then we can use successively straightening transformations including the straightening of some orbit segments such that system (1.7) has the following C^k normal in U_i: as $z = \left(x, y, v\right)^* \in U_i, i = 1,3$

$$\begin{aligned} \dot{x} &= \left[\lambda_i^1\left(\mu\right) + o\left(1\right)\right]x, \\ \dot{y} &= \left[-\rho_i^1\left(\mu\right) + o\left(1\right)\right]y + \mathrm{O}\left(v\right)\left(\mathrm{O}\left(x\right) + \mathrm{O}\left(v\right)\right), \\ \dot{v} &= \left[-\rho_i^2\left(\mu\right) + o\left(1\right)\right]v + \mathrm{O}\left(y\right)\left(\mathrm{O}\left(x\right) + \mathrm{O}\left(y\right)\right), \end{aligned} \tag{1.8}$$

and as $z = \left(x, y, u\right)^* \in U_2$

$$\begin{aligned} \dot{x} &= \left[\lambda_2^1\left(\mu\right) + o\left(1\right)\right]x + \mathrm{O}\left(u\right)\left(\mathrm{O}\left(y\right) + \mathrm{O}\left(u\right)\right) \\ \dot{y} &= \left[-\rho_2^1\left(\mu\right) + o\left(1\right)\right]y, \\ \dot{u} &= \left[\lambda_2^1\left(\mu\right) + o\left(1\right)\right]u + \mathrm{O}\left(x\right)\left(\mathrm{O}\left(x\right) + \mathrm{O}\left(y\right)\right), \end{aligned} \tag{1.9}$$

where $k \geq r - 2$, the sign "$*$" stands for transposition. For $\|u\|$ sufficiently small, where $\lambda_i^1\left(\mu\right) = \lambda_i^1$, $\rho_i^1\left(\mu\right) = \rho_i^1$, $\rho_i^2\left(\mu\right) = \rho_i^2\left(i = 1,3\right)$, $\lambda_2^1\left(\mu\right) = \lambda_2^1$, $\lambda_2^1\left(\mu\right) = \lambda_2^1$, $\rho_2^1\left(\mu\right) = \rho_2^1$ is the corresponding eigenvalues of the linearization matrix of perturbed system (1.6).

(H₂) (*non-degeneration*) System (1.7) has a double heterodimensional cycles $\gamma = \gamma_1\left(t\right) \cup \gamma_2\left(t\right) \cup \gamma_3\left(t\right) \cup \gamma_4\left(t\right)$, where $\Gamma_i = \left\{z = \gamma_i\left(t\right): t \in R\right\}$,
$\gamma_1\left(+\infty\right) = r_2\left(-\infty\right) = p_1$, $\gamma_1\left(+\infty\right) = \gamma_2\left(-\infty\right) = \gamma_3\left(-\infty\right) = \gamma_4\left(+\infty\right) = p_2$,
$\gamma_3\left(+\infty\right) = \gamma_4\left(-\infty\right) = p_3$, and

$$\dim\left(T_{\gamma_1(t)}W_1^u \cap T_{\gamma_1(t)}W_2^s\right) = 1, \quad \dim\left(T_{\gamma_2(t)}W_2^u \cap T_{\gamma_2(t)}W_1^s\right) = 1,$$

$$\dim\left(T_{\gamma_3(t)}W_2^u \cap T_{\gamma_3(t)}W_3^s\right) = 1, \quad \dim\left(T_{\gamma_4(t)}W_3^u \cap T_{\gamma_4(t)}W_2^s\right) = 1,$$

Here $\gamma_i\left(t\right)$ represents the flow of system (17), $t \in R$ and by $T_q M$ we denote the tangent space of the manifold *M* at *q*.

(H₃) (*Orbit flip*) Let $e_i^{\mp} = \lim_{t \to \mp\infty} \dot{\gamma}_i\left(-t\right) / \left|\dot{\gamma}_i\left(-t\right)\right|$, then

$$e_1^+ \in T_{p_1}W_1^u, \quad e_2^+, e_3^+ \in T_{p_2}W_2^u, \quad e_4^+ \in T_{p_3}W_3^u,$$

$$e_1^-, e_4^- \in T_{p_2}W_2^s, \quad e_2^- \in T_{p_1}W_1^s, \quad e_3^- \in T_{p_3}W_3^u,$$

where $e_i^+, e_i^-\left(i = 1,2,3\right)$ are unit eigenvectors corresponding to λ_i^1 and $\rho_i^1\left(i = 1,2,3\right)$ respectively. Furthermore they satisfy the equation $e_1^- = -e_4^-$, $e_3^+ = -e_2^+$ (for details see [19]).

Here, e_2^+ and e_2^- are the unit eigenvectors corresponding to λ_2^1 and $-\rho_1^2$

which responds Γ_2 enters the equilibrium p_1 along the strong stable manifold W_1^{ss} (as $t \to +\infty$, enters the equailibruium p_2 along the unstable manifold W_2^u (as $t \to -\infty$), that is, from [17], the heteroclinic orbit Γ_2 has orbits flips when $t \to +\infty$ (see **Figure 1**).

(H_4) (*Strong inclination*)

$$\lim_{t\to+\infty} T_{\gamma_1(t)}W_1^u = span\left\{e_1^-, e_2^+\right\}, \quad \lim_{t\to-\infty} T_{\gamma_1(t)}W_2^s = span\left\{e_1^+, e_2^-\right\};$$
$$\lim_{t\to+\infty} T_{\gamma_2(t)}W_2^u = span\left\{e_2^-\right\}, \quad \lim_{t\to-\infty} T_{\gamma_2(t)}W_1^s = span\left\{e_2^+\right\};$$
$$\lim_{t\to+\infty} T_{\gamma_3(t)}W_2^u = span\left\{e_3^-\right\}, \quad \lim_{t\to-\infty} T_{\gamma_3(t)}W_3^s = span\left\{e_3^+\right\};$$
$$\lim_{t\to+\infty} T_{\gamma_4(t)}W_3^u = span\left\{e_4^-, e_3^+\right\}, \quad \lim_{t\to-\infty} T_{\gamma_4(t)}W_2^s = span\left\{e_4^+, e_3^-\right\}.$$

Remark 1.1. *Under the assumption* H_1, p_1 *and* p_3 *have a* 1*-dimensional unstable manifold and a* 2*-dimensional stable manifold, while* p_2 *has a* 2*-dimensional unstable manifold and a* 1*-dimensional stable manifold, hence* Γ *is double heterodimensional cycles.*

Remark 1.2. *Hypothesis* (H_4) *shows that* $W_{p_i}^u$ *and* $W_{p_i}^s$ *have strong inclination property. Due to the assumption* (H_2), p_2 *has a* 2*-dimensional unstable manifold,* p_3 *has a* 2*-dimensional stable manifold, and* $\dim\left(T_{\gamma_3(t)}W_3^u \cap T_{\gamma_2(t)}W_2^s\right) = 1$, *we can know the codimension of the heteroclinic orbit* Γ_3 *is* 0. *Then the orbits* Γ_3 *is transversal, that is, they can be preserved even under small perturbations.*

2. Local Coordinates and Bifurcation Equations

In this section, we need first to take fundamental solutions of linear variational Equation (see Equation (1.6) as below) and use them as an active coordinate system along the heteroclinic orbits. Then using the new coordinates, we construct the global map spanned by the flow of (1.6) between the sections along the orbits. Next, we set up local maps near equilibriums. Finally the whole Poincaré map can be obtained by composing these maps. The implicit function theorem reveals the bifurcation equation.

By the stable and unstable manifolds theorem and up to two local linear transformations, we see that there are three open neighborhoods U_i of $p_i = (0,0,0)^*$

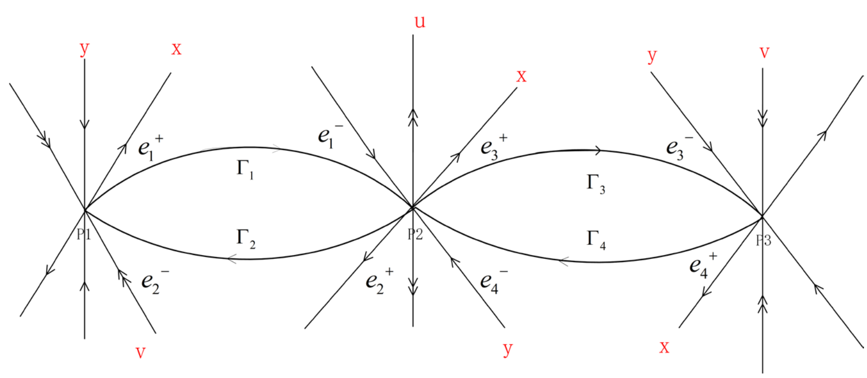

Figure 1. Double heterodimensional cycles of three saddle points p_i with four orbits $\gamma_k(t)$.

such that p_i have C^{r-1} local manifolds $W^s_{i,loc}$ and $W^s_{i,loc}\left(i=1,2,3\right)$ which are expressed as below: for $j=1,3$,

$$W^u_{j,loc}=\left\{z=\left(x,y,v\right)^*\in U_j \mid \left(y,v\right)=\left(y,v\right)\left(x\right),\left(y,v\right)\left(0\right)=0,\frac{\partial\left(y,v\right)}{\partial x}\left(0\right)=0\right\},$$

$$W^s_{j,loc}=\left\{z=\left(x,y,v\right)^*\in U_j \mid x=x\left(y,v\right),x\left(0,0\right)=0,\frac{\partial x}{\partial\left(y,v\right)}\left(0,0\right)=0\right\},$$

$$W^u_{2,loc}=\left\{z=\left(x,y,u\right)^*\in U_2 \mid y=y\left(x,u\right),y\left(0,0\right)=0,\frac{\partial y}{\partial\left(x,u\right)}\left(0,0\right)=0\right\},$$

$$W^s_{2,loc}=\left\{z=\left(x,y,u\right)^*\in U_2 \mid \left(x,u\right)=\left(x,u\right)(y),\left(x,u\right)\left(0\right)=0,\frac{\partial\left(x,u\right)}{\partial y}\left(0\right)=0\right\}.$$

Let the coordinate expression of $\gamma_k\left(t\right)$ be $\gamma_k\left(t\right)=\left(\gamma_k^x\left(t\right),\gamma_k^y\left(t\right),\gamma_k^v\left(t\right)\right)^*$ in the small neighborhood U_i of p_i, $\left(i=1,3\right)$, and $\gamma_k\left(t\right)=\left(\gamma_k^x\left(t\right),\gamma_k^y\left(t\right),\gamma_k^u\left(t\right)\right)^*$ in the small neighborhood U_2 of p_2. Since $T_k>0\left(k=1,2,3,4\right)$ is large enough so that $\gamma_1\left(-T_1\right),\gamma_2\left(T_2\right)\in U_1$, $\gamma_3\left(T_3\right),\gamma_4\left(-T_4\right)\in U_3$, $\gamma_1\left(T_1\right),\gamma_2\left(-T_2\right),\gamma_3\left(-T_3\right),\gamma_4\left(T_4\right)\in U_2$ and for $k=1,3,4$, $\gamma_k\left(-T_k\right)=\left(\delta,0,0\right)^*$, $\gamma_2\left(-T_2\right)=\left(-\delta,0,0\right)^*$, for $k=3,4$, $\gamma_k\left(T_k\right)=\left(0,\delta,0\right)$, $r_1\left(T_1\right)=\left(0,0,\delta\right)$, $r_2\left(T_2\right)=\left(0,-\delta,0\right)$, where $\delta>0$ is small enough.

Now we take into account the linearly variational system and its corresponding adjoint system of (1.7) formed respectively by: let $A_k\left(t\right)=Df\left(\gamma_k\left(t\right)\right)$,

$$\dot z=A_k\left(t\right)z \tag{2.1}$$

and

$$\dot\phi=-A_k\left(t\right)^*\phi \tag{2.2}$$

Based on the above hypotheses about system (1.7), system (2.1) has exponential dichotomies in $\mathbf{R}^+$ and $\mathbf{R}^-$ (see [12]). We can obtain the following properties.

Lemma 2.1. *System* (2.1) *has the fundamental solution matrices*

$$Z_k\left(t\right)=\left(z_k^1\left(t\right),z_k^2\left(t\right),z_k^3\left(t\right)\right)\left(k=1,2,3,4\right)$$

which satisfy, respectively, for $k=1,4$

$$z_k^1\left(t\right),z_k^3\left(t\right)\in\left(T_{\gamma_k(t)}\Gamma_k\left(\mu\right)\right)^c,$$
$$z_k^2\left(t\right)=\gamma_k\left(t\right)/\left|\gamma_k\left(T_k\right)\right|\in T_{\gamma_k(t)}W_k^u\cap T_{\gamma_k(t)}W^s_{k-(-1)^k}$$

that is

$$Z_k\left(-T_k\right)=\begin{pmatrix}0 & w_k^{21} & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix},\quad Z_k\left(T_k\right)=\begin{pmatrix}w_k^{11} & 0 & w_k^{31}\\ w_k^{12} & \left(-1\right)^k & w_k^{32}\\ w_k^{13} & 0 & w_k^{33}\end{pmatrix} \tag{2.3}$$

where $w_k=\begin{vmatrix}w_k^{11} & w_k^{31}\\ w_k^{13} & w_k^{33}\end{vmatrix}\neq0,\left|w_k^{i2}\cdot w_k^{-1}\right|\ll1,i=1,3,W_4^u=W_2^u$.

And for $k = 2,3$

$$z_k^1(t) \in \left(T_{\gamma_k(t)} W_2^u\right)^c \cap T_{\gamma_k(t)} W_{k-(-1)^k}^s,$$
$$z_k^2(t) = \dot{r}_k(t) / \left|\dot{\gamma}_k(-T_k)\right| \in T_{\gamma_k(t)} W_2^u \cap T_{\gamma_k(t)} W_{k-(-1)^k}^s,$$
$$z_k^3(t) \in T_{\gamma_k(t)} W_2^u \cap \left(T_{\gamma_k(t)} W_{k-(-1)^k}^s\right)^c,$$

that is,

$$Z_k(-T_k) = \begin{pmatrix} 0 & w_k^{21} & 0 \\ \overline{w}_k^{12} & 0 & 1 \\ 1 & 0 & w_k^{33} \end{pmatrix}$$
$$Z_2(T_2) = \begin{pmatrix} 1 & 0 & w_2^{31} \\ w_2^{12} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.4}$$
$$Z_3(T_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w_3^{22} & w_3^{32} \\ w_3^{13} & 0 & 1 \end{pmatrix}$$

where $w_2^{21} < 0, w_3^{22} \neq 0, \left|w_2^{12} \cdot w_2^{31}\right| \ll 1, \left|w_3^{13} \cdot w_3^{32}\right| \ll 1, \left|w_k^{33} \cdot \left(w_k^{21}\right)^{-1}\right| \ll 1, W_4^s = W_3^u$.

In what follows, we select $\left(Z_k^2(t), Z_k^2(t), Z_k^3(t)\right)(k = 1,2,3,4)$ as a new local coordinate system along Γ_k. Let $\theta_k(t) = \left(\phi_1(t), \phi_2(t), \phi_3(t)\right) = \left(Z^{-1}(t)\right)^*$ be the fundamental solution matrix of (2.2). By the [1, ?], we can know that the $\phi_k^1(t)$ is bounded and tends to zero exponentially as $t \to \pm\infty$.

Take a coordinate transformation

$$z(t) = h_k(t) = \gamma_k(t) + Z_k(t) N_k(t), t \in [-T_k, T_k], 0 < \varepsilon \ll \delta. \tag{2.5}$$

in a small neighborhood of Γ_k, where $N_k(t) = \left(n_k^1(t), 0, n_k^3(t)\right)^*, k = 1,2,3,4$, and $n_k^1(t), n_k^3(t)$ represents the coordinate decomposition of (1.6) in the new local coordinate system corresponding to $Z_k^1(t)$ and $Z_k^3(t)$, Then we can take eight transverse sections vertical to the tangency $T_{\gamma_k(t)}$ to each orbit $\gamma_k(t)$ (see Figure 2)

$$S_1^0 = \left\{z = h_1(-T_1) : |x|, |y - \delta|, |v - \delta| < \varepsilon\right\},$$
$$S_2^0 = \left\{z = h_2(-T_2) : -|x|, |y - \delta|, |u - \delta| < \varepsilon\right\},$$

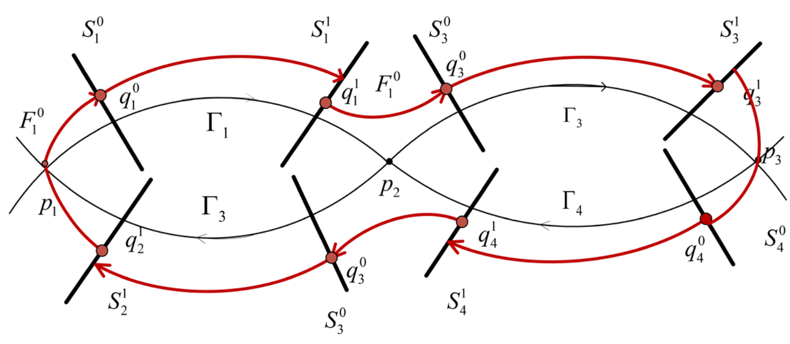

Figure 2. The cross-section and Poincaré map.

$$S_3^0 = \{z = h_3(-T_1) : |x|, |y-\delta|, |u-\delta| < \varepsilon\},$$
$$S_4^0 = \{z = h_4(-T_4) : |x|, |y-\delta|, |v-\delta| < \varepsilon\},$$
$$S_1^1 = \{z = h_1(T_1) : |x-\delta|, -|y|, |u-\delta| < \varepsilon\},$$
$$S_2^1 = \{z = h_2(T_2) : |x-\delta|, |y-\delta|, |v| < \varepsilon\},$$
$$S_3^1 = \{z = h_3(T_3) : |x-\delta|, |y|, |v-\delta| < \varepsilon\},$$
$$S_4^1 = \{z = h_4(T_4) : |x-\delta|, |y|, |u-\delta| < \varepsilon\}$$

In order to obtain the corresponding bifurcation equation, we need to restrict our attention to set up the Poincaré return map of system (1.6). Firstly, we find the relationship between the old coordinates

$$q_k^0\left(x_k^0, y_k^0, \bar{u}_k^0\right), \quad q_j^1\left(x_j^0, y_j^0, \bar{u}_k^o\right)$$

and new coordinates

$$q_i^0\left(n_i^{0,1}, 0, n_i^{0,3}\right), \quad q_i^1\left(n_i^{1,1}, 0, n_i^{1,3}\right)$$

where $k = 2,3, j = 1,4, \bar{u}_k^0 = u_k^0$; $k = 1,4, j = 2,3, \bar{u}_k^0 = v_k^0$. Then, combining with the Equations (2.3), (2.4), we obtain for $k = 1,4$

$$\begin{cases} n_k^{0,1} = v_k^0 \\ n_k^{0,3} = y_k^0 \\ x_k^0 = \delta \end{cases} \tag{2.6}$$

and

$$\begin{cases} n_k^{1,1} = w_k^{-1}\left(w_k^{33}x_k^1 - w_k^{31}u_k^1\right) \\ n_k^{1,3} = w_k^{-1}\left(w_k^{11}u_k^1 - w_k^{13}x_k^1\right) \\ y_k^1 = \delta + w_k^{-1}\left(w_k^{12}w_k^{33} - w_k^{32}w_k^{13}\right)x_k^1 + w^{-1}\left(w_k^{31}w_k^{11} - w_k^{12}w_k^{31}\right)u_k^1 \approx \delta \end{cases} \tag{2.7}$$

for $k = 2,3$

$$\begin{cases} n_k^{0,1} = u_k^0 - w_k^{33}y_k^0 \\ n_k^{0,3} = y_k^0 - w_k^{12}u_k^0 \\ x_k^0 = (-1)^{k-1}\delta \end{cases} \tag{2.8}$$

and

$$\begin{cases} n_2^{1,1} = x_2^1 - w_2^{31}y_2^1 \\ n_2^{1,3} = y_2^1 - w_2^{12}x_2^1 \\ v_2^1 = \delta \end{cases}, \begin{cases} n_3^{1,1} = x_3^1 - w_3^{32}v_3^1 \\ n_3^{1,3} = v_3^1 - w_3^{13}x_3^1 \\ y_3^1 = \delta \end{cases} \tag{2.9}$$

Then, under transformation (2.5), system (1.6) has the following form by $\dot{\gamma}(t) = f(\gamma(t))$ and $\dot{Z}_k(t) = Df(\gamma(t))Z_k(t)$:

$$\dot{N}_k(t) = \theta_k^*(t)\, g_\mu(\gamma_k(t), 0)\mu + h.o.t, \tag{2.10}$$

where g_μ is the partial derivation of $g(z,\mu)$ with respect to μ. To integrate (2.10), we get

$$\begin{aligned} N_k\left(T_k\right) &= N_k\left(-T_k\right)+\int_{-T_k}^{T_k} \theta_k^*\left(t\right) g_\mu\left(\gamma_k\left(t\right),0\right)\mu \mathrm{d}t + h.o.t. \\ &\triangleq N_k\left(-T_k\right)+M_k^j\mu + h.o.t. \end{aligned} \tag{2.11}$$

where $M_k^j\left(\mu\right)=\int_{-T_k}^{T_k}\theta_k^{j*}\left(t\right)g_\mu\left(\gamma_k\left(t\right),0\right)\mu\mathrm{d}t\ \left(j=1,3;k=1,2,3,4\right)$ are called Melnikov vectors respect to μ.

Which are defined as the global maps $F_k^1: S_k^0 \to S_k^1\left(k=1,2,3,4\right)$ with the expression by (2.11) given

$$\begin{aligned} \bar{n}_k^{1,1} &= n_k^{0,1}+M_k^1\mu + h.o.t., \\ \bar{n}_k^{1,3} &= n_k^{0,3}+M_k^3\mu + h.o.t.. \end{aligned} \tag{2.12}$$

as follows

$$F_k^1\left(n_k^{0,1},0,n_k^{0,3}\right)=\left(\bar{n}_k^{1,1},0,\bar{n}_k^{1,3}\right).$$

Next we consider the local maps,

$$\begin{aligned} &F_1^0: q_2^1\in S_2^1 \mapsto q_1^0 \in S_1^0, \quad F_2^0: q_1^1\in S_1^1 \mapsto q_3^0 \in S_3^0, \\ &F_3^0: q_3^1\in S_3^1 \mapsto q_4^0 \in S_4^0, \quad F_4^0: q_4^1\in S_4^1 \mapsto q_2^0 \in S_2^0 \end{aligned}$$

induced by flows confined in the neighborhood $U_i\left(i=1,2,3\right)$.

Let τ_1,τ_3 be the time spent from q_2^1 to q_1^0 and from q_3^1 to q_4^0 respectively, corresponding their Shilnikov time $s_1=\mathrm{e}^{-\rho_1^1(\mu)\tau_1}, s_3=\mathrm{e}^{-\rho_3^1(\mu)\tau_3}$. Let τ_2,τ_4 be the time spent from q_1^1 to q_3^0 and from q_4^1 to q_2^0, then their Shilnikov time are $s_k=\mathrm{e}^{\lambda_2^1(\mu)\tau_k}\left(k=2,4\right)$.

Then under the assumptions among the eigenvalues, by the normal forms (1.8)-(1.9), and the formula of variation of constants, we obtain the local maps:

$$\begin{aligned} &F_1^0: q_2^1\left(x_2^1,y_2^1,v_2^1\right)\in S_2^1 \mapsto q_1^0\left(x_1^0,y_1^0,v_1^0\right)\in S_1^0 \\ &x_2^1 = x\left(T_2\right)\approx x_1^0 s_1^{\frac{\lambda_1^1(\mu)}{\rho_1^1(\mu)}}, \quad y_1^0=y\left(T_2+\tau_1\right)\approx y_2^1 s_1, \quad v_1^0 = v\left(T_2+\tau_2\right)\approx v_2^1 s_1^{\frac{\rho_1^2(\mu)}{\rho_1^1(\mu)}}. \end{aligned} \tag{2.13}$$

$$\begin{aligned} &F_2^0: q_1^1\left(x_1^1,y_1^1,u_1^1\right)\in S_1^1 \mapsto q_3^0\left(x_3^0,y_3^0,u_3^0\right)\in S_3^0 \\ &x_1^1 = x\left(T_1\right)\approx x_3^0 s_2, \quad y_3^0=y\left(T_1+\tau_2\right)\approx y_1^1 s_2^{\frac{\rho_2^1}{\lambda_2^1}}, \quad u_1^1 = u\left(T_2\right)\approx u_3^0 s_2^{\frac{\lambda_2^2(\mu)}{\lambda_2^1(\mu)}}. \end{aligned} \tag{2.14}$$

$$\begin{aligned} &F_3^0: q_3^1\left(x_3^1,y_3^1,v_3^1\right)\in S_3^1 \mapsto q_4^0\left(x_4^0,y_4^0,v_4^0\right)\in S_4^0 \\ &x_3^1 = x\left(T_3\right)\approx x_4^0 s_3^{\frac{\lambda_3^1(\mu)}{\rho_3^1(\mu)}}, \quad y_4^0=y\left(T_3+\tau_3\right)\approx y_3^1 s_3, \quad v_4^0 = v\left(T_3+\tau_3\right)\approx v_3^1 s_3^{\frac{\rho_3^2(\mu)}{\rho_3^1(\mu)}} \end{aligned} \tag{2.15}$$

$$\begin{aligned} &F_4^0: q_4^1\left(x_4^1,y_4^1,u_4^1\right)\in S_4^1 \mapsto q_2^0\left(x_2^0,y_2^0,u_2^0\right)\in S_2^0 \\ &x_4^1 = x\left(T_4\right)\approx x_2^0 s_4, \quad y_2^0=y\left(T_4+\tau_4\right)\approx y_4^1 s_4^{\frac{\rho_2^1(\mu)}{\lambda_2^1(\mu)}}, \quad u_4^1 = u\left(T_4\right)\approx u_2^0 s_4^{\frac{\lambda_2^2(\mu)}{\lambda_2^1(\mu)}} \end{aligned} \tag{2.16}$$

Thus, by (2.6), (2.12) (2.13), we obtain the first Poincaré map $F_1 = F_1^1\circ F_1^0: S_2^1\mapsto S_1^1$ as follows

$$\begin{cases} \bar{n}_1^{1,1} = M_1^1\mu + h.o.t. \\ \bar{n}_1^{1,3} = M_1^3\mu + h.o.t.; \end{cases} \tag{2.17}$$

by (2.8), (2.12), (2.14), we obtain the Poincaré map $F_3 = F_3^1\circ F_2^0: S_1^1\mapsto S_3^1$ as follows

$$\begin{cases}\overline{n}_3^{1,1}=u_3^0+w_3^{33}\delta s_2^{\beta_2}+M_3^1\mu+h.o.t.\\ \overline{n}_3^{1,3}=w_3^{12}u_3^0+\delta s_2^{\beta_2}+M_3^3\mu+h.o.t.\end{cases} \tag{2.18}$$

by (2.8), (2.12), (2.15), we obtain the Poincaré map $F_4=F_4^1\circ F_3^0:S_3^1\mapsto S_4^1$ as follows

$$\begin{cases}\overline{n}_4^{1,1}=M_4^1\mu+h.o.t.\\ \overline{n}_4^{1,3}=M_4^3\mu+h.o.t.\end{cases} \tag{2.19}$$

by (2.6), (2.12), (2.16), we obtain the Poincaré map $F_2=F_2^1\circ F_4^0:S_4^1\mapsto S_2^1$ as follows

$$\begin{cases}\overline{n}_2^{1,1}=u_2^0-w_2^{33}\delta s_4^{\beta_2}+M_2^1\mu+h.o.t.\\ \overline{n}_2^{1,3}=\delta s_4^{\beta_2}-w_2^{12}u_2^0+M_2^3\mu+h.o.t.\end{cases} \tag{2.20}$$

Then, by (2.7), (2.9), (2.17), (2.18), (2.19), (2.20), we induce the successor functions

$$\begin{aligned}G&=\left(G_1,G_2,G_3\right)=G\left(G_1^1,G_1^3,G_3^1,G_3^3,G_4^1,G_4^3,G_2^1,G_2^3\right)\\ &=G\left(s_1,s_2,s_3,s_4,u_1^0,u_4^0,v_1^1,v_4^1\right)\\ &=\left(F_1\left(q_2^1\right)-q_1^1,F_3\left(q_1^1\right)-q_3^1,F_2\left(q_3^1\right)-q_4^1,F_2\left(q_4^1\right)-q_2^1\right)\end{aligned}$$

where

$$G_1^1=w_1^{-1}w_1^{33}\delta s_2^{\frac{\lambda_2^2}{\lambda_2^1}}+w_1^{-1}w_1^{31}u_3^0s_2+M_1^1\mu+h.o.t.;$$

$$G_1^3=w_1^{-1}w_1^{11}u_3^0s_2+w_1^{-1}w_1^{13}\delta s_2^{\frac{\lambda_2^2}{\lambda_2^1}}+M_1^3\mu+h.o.t.$$

$$G_2^1=u_2^0-w_2^{33}\delta s_4^{\beta_2}+w_2^{31}y_2^1+M_2^1\mu+h.o.t.;$$
$$G_2^3=\delta s_4^{\beta_2}-w_2^{12}u_2^0-y_2^1+M_2^3\mu+h.o.t.$$

$$G_3^1=u_3^0+w_3^{33}\delta s_2^{\beta_2}+w_3^{32}v_3^1+M_3^1\mu+h.o.t.;$$
$$G_3^3=w_3^{12}u_3^0+\delta s_2^{\beta_2}-v_3^1+M_3^3\mu+h.o.t.$$

$$G_4^1=w_4^{-1}w_4^{33}\delta s_4^{\frac{\lambda_2^2}{\lambda_2^1}}+w_4^{-1}w_4^{31}u_2^0s_4+M_4^1\mu+h.o.t.;$$

$$G_4^3=-w_4^{-1}w_4^{11}u_2^0s_4+w_4^{-1}w_4^{13}\delta s_4^{\frac{\lambda_2^2}{\lambda_2^1}}+M_4^3\mu+h.o.t.$$

By the implicit function theorem, solving the equation $\left(G_2^1,G_2^3,G_3^1,G_3^3\right)=0$, we have

$$\begin{aligned}u_2^0&=\left(w_2^{31}-w_2^{33}\right)\delta s_4^{\beta_2}-w_2^{31}M_2^3\mu-M_2^1\mu+h.o.t.\\ y_2^1&=\delta s_4^{\beta_2}-w_2^{12}M_2^1\mu-M_2^3\mu+h.o.t.\\ u_3^0&=-\left(w_3^{32}+w_3^{33}\right)\delta s_2^{\beta_2}-w_3^{32}M_3^3\mu-M_3^1\mu+h.o.t.\\ v_3^1&=\delta s_2^{\beta_2}-w_3^{12}M_3^1\mu+M_3^3\mu+h.o.t.\end{aligned}$$

Substituting them into $\left(G_4^1,G_4^3,G_1^1,G_1^3\right)=0$, we obtain the bifurcation equations, for $w_1^{13}\neq 0,w_1^{33}\neq 0,w_4^{13}\neq 0,w_4^{33}\neq 0$

$$\begin{cases} w_4^{33}\delta s_4^{\frac{\lambda_2^2}{\lambda_2^1}} - w_4^{31}s_4\left(w_2^{31}M_2^3\mu + M_2^1\mu\right) + w_4M_4^1\mu + h.o.t. \\ w_4^{13}\delta s_4^{\frac{\lambda_2^2}{\lambda_2^1}} + w_4^{11}s_4\left(w_2^{31}M_2^3\mu + M_2^1\mu\right) + w_4M_4^3\mu + h.o.t. \\ w_1^{33}\delta s_2^{\frac{\lambda_2^2}{\lambda_2^1}} - w_1^{31}s_2\left(w_3^{32}M_3^3\mu + M_3^1\mu\right) + w_1M_1^1\mu + h.o.t. \\ w_1^{13}\delta s_2^{\frac{\lambda_2^2}{\lambda_2^1}} - w_1^{11}s_2\left(w_3^{32}M_3^3\mu + M_3^1\mu\right) + w_1M_1^3\mu + h.o.t. \end{cases} \tag{2.21}$$

Remark 2.1. *In fact,* M_i^j *is independent of the choice of* T_i *for* $i = 1,2,3,4,\ j = 1,3$, *which can be verified similarly as in* [13]

Remark 2.2. *Generally, in two-dimensional plane system, when we study bifurcations of singular cycle, Poincaré mapping can only be established on one side of the singular cycle. Therefore, there are no other types of orbits except the one with infinite approaching to saddle point on the left side of* p_1 *and the right side of* p_3. *However, in high-dimensional system, it remains to be verified whether other types of orbits can bypass different surfaces for connection. To make the study go on, we assume that* $s_1 = s_3 = 0$, *that is, the orbit starting from* S_2^1 *to* S_1^0 *just be a singular orbit which is infinitely approaching* p_1 *when* $t \to \pm\infty$; *for the orbit starting from* S_3^1 *to* S_4^0 *is similar near* p_3.

Remark 2.3. *Basing on remark* 2.2, *it can be seen that* (2.13) *and* (2.14) *become* $F_1^0 : q_2^1\left(0, y_2^1, \delta\right) \in S_2^1 \mapsto q_1^0\left(\delta, 0, 0\right) \in S_1^0$ *and* $F_3^0 : q_3^1\left(0, \delta, v_3^1\right) \in S_3^1 \mapsto q_4^0\left(\delta, 0, 0\right) \in S_4^0$.

Remark 2.4. *Shilnikov variables were introduced by Shilnikov in 1968 to compute the local transition map near equilibria to leading order. Instead of solving an initial-value problem, solutions near the equilibrium are found using an appropriate boundary-value problem.*

3. Heterodimensional Cycle Bifurcation of "∞" Type

In this section, we analyze the bifurcation of system (1.6) under hypotheses (A_1)-(A_4). The existence of "∞"-shape double heterodimensional cycles, the heteroclinic cycle composed of three orbits and connecting with three saddle points, and large 1-heteroclinic connecting with p_1 and p_3 are studied by discussing the corresponding bifurcation equation. Clearly if $s_2 = s_4 = 0$ the double heterodimensional cycle ("∞") of system (1.6) is persistent; if $s_4 = 0$, $s_2 > 0$, system (1.6) has a heterodimensional cycle consisting of two saddles of (1.2) type and one saddle of (2.1) type composed of one big orbit linking p_3, p_1 and two orbits linking p_3, p_2 and p_2, p_1 respectively, which is called the second shape heterodimensional cycle in later of this paper; if $s_2 = 0$, $s_4 > 0$, system (1.6) has another heterodimensional cycle consisting of two saddles of (2.1) type and one saddle of (1.2) type composed of one big orbit linking p_1, p_3 and two orbits linking p_1, p_2 and p_2, p_3 respectively, which is called another second shape heterodimensional cycle in later of this paper; if $s_2 > 0$ and $s_4 > 0$, system (1.6) has the large 1-heteroclinic cycle consisting of two saddles

p_1 and p_3 of (2.1) type composed with two big orbits linking p_1, p_3 and p_3, p_1 respectively. What is noteworthy is that if the conditions make $s_2 > 0$ untenable and set $s_4 = 0$ tenable, the conditions make $s_4 > 0$ untenable and $s_2 = 0$ tenable, system (1.6) has the third heterodimensional cycle consisting of one saddle p_2 of (2.1) type and one saddle p_3 of (1.2) type and composed of one orbit starting from p_2 to p_3 and another orbit starting from p_3 to p_2 under the assumption (H_2). So in the following, we need to consider solutions s_2 and s_4 of the bifurcation Equation (2.21).

3.1. Analysis Procedure

Corresponding results about the existence of the second heterodimensional cycle, the third heterodimensional cycle and large-1 heteroclinic cycle, as well as the coexistence of double heterodimensional cycle and the large 1-heteroclinic cycle are contained in the next theorems. For convenience to discuss, we set eight regions:

$$N_+^+ = \left\{\mu \mid w_1^{31}\left(w_3^{32} M_3^3 \mu + M_3^1 \mu\right) > 0, w_1 M_1^1 \mu < 0\right\},$$
$$N_+^- = \left\{\mu \mid w_1^{31}\left(w_3^{32} M_3^3 \mu + M_3^1 \mu\right) > 0, w_1 M_1^1 \mu > 0\right\}$$
$$N_-^+ = \left\{\mu \middle| w_1^{31}\left(w_3^{32} M_3^3 \mu + M_3^1 \mu\right) < 0, w_1 M_1^1 \mu < 0\right\},$$
$$N_-^- = \left\{\mu \middle| w_1^{31}\left(w_3^{32} M_3^3 \mu + M_3^1 \mu\right) < 0, w_1 M_1^1 \mu > 0\right\};$$
$$B_+^+ = \left\{\mu \middle| w_4^{31}\left(w_2^{32} M_2^3 \mu + M_2^1 \mu\right) > 0, w_4 M_4^1 \mu < 0\right\},$$
$$B_+^- = \left\{\mu \middle| w_4^{31}\left(w_2^{32} M_2^3 \mu + M_2^1 \mu\right) > 0, w_4 M_4^1 \mu > 0\right\},$$
$$B_-^+ = \left\{\mu \middle| w_4^{31}\left(w_2^{32} M_2^3 \mu + M_2^1 \mu\right) < 0, w_4 M_4^1 \mu < 0\right\},$$
$$B_-^- = \left\{\mu \middle| w_4^{31}\left(w_2^{32} M_2^3 \mu + M_2^1 \mu\right) < 0, w_4 M_4^1 \mu > 0\right\}$$

From the discussion of Theorem 1, if one of s_2 and s_4 is 0, the second heterodimensional cycle will appear. And if $s_2 > 0$, $s_4 > 0$, a large 1-heteroclinic cycle connecting with p_1 and p_3 will exist. As well as, if there are conditions that make $s_2 > 0$ be invalid and $s_4 = 0$ or $s_4 > 0$ be invalid and $s_2 = 0$, the third heterodimensional cycle will arise. Therefore it is enough to discuss the solutions $s_i\left(i = 2, 4\right)$ of the Equation (2.21).

Since the first two equations of Equation (2.11) have the same structure as the last two, we only analyze the first and second equations as following

$$\begin{cases} w_4^{33}\delta s_4^{\frac{\lambda_2^2}{\lambda_2^1}} - w_4^{31} s_4\left(w_2^{31} M_2^3 \mu + M_2^1 \mu\right) + w_4 M_4^1 \mu + h.o.t. = 0 \\ w_4^{13}\delta s_4^{\frac{\lambda_2^2}{\lambda_2^1}} + w_4^{11} s_4\left(w_2^{31} M_2^3 \mu + M_2^1 \mu\right) + w_4 M_4^3 \mu + h.o.t. = 0 \end{cases} \tag{3.1}$$

Set $\alpha = \dfrac{\lambda_2^2}{\lambda_2^1}$, rewrite the first Equation of (3.1) as

$$w_4^{31}\left(w_2^{31} M_2^3 \mu + M_2^1 \mu\right) s_4 - w_4 M_4^1 \mu - w_4^{33}\delta s_4^{\alpha} + h.o.t. \triangleq L\left(s_4, \mu\right) - N\left(s_4, \mu\right) = 0, \tag{3.2}$$

where

$$L(s_4,\mu)=w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)s_4-w_4M_4^1\mu,\quad N(s_4,\mu)=w_4^{33}\delta s_4^{\alpha}+h.o.t..$$

Then we have

$$L(0,\mu)-N(0,\mu)=-w_4M_4^1\mu+h.o.t.,$$

$$L'_{s_4}(s_4,\mu)-N'_{s_4}(s_4,\mu)=w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)-w_4^{33}\alpha\delta s_4^{\alpha-1}+h.o.t..$$

If $\left(w_4^{33}\right)^{-1}w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)>0$, the equation $L'_{s_4}(s_4,\mu)-N'_{s_4}(s_4,\mu)=0$ has a unique small positive solution $\tilde{s}_4=\left(w_4^{31}\left(w_4^{33}\alpha\delta\right)^{-1}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right)^{\frac{1}{\alpha-1}}+h.o.t.$ If $\left(w_4^{33}\right)^{-1}w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)<0$, it makes $\tilde{s}_4>0$ be untenable.

1) If $w_4^{33}>0$, $\mu\in B_-^-$ or $w_4^{33}<0$, $\mu\in B_+^+$, the straight line L and the curve N cannot intersect in the half plane for $s_4>0$, so Equation (3.2) has not any positive solutions, that is, system (1.6) only has the transversal heteroclinic orbit $\gamma_3(t)$ in the region $\Gamma_2\cup\Gamma_3$.

2) If $w_4^{33}>0$, $\mu\in B_-^+$ or $w_4^{33}<0$, $\mu\in B_+^-$, the straight line $L(t_4,\mu)$ and the curve $N(s_4,\mu)$ intersect at one positive point, that is, (3.2) has one positive solution.

Without loss of generality, we discuss the case $w_4^{33}>0$, $\mu\in B_-^+$. There are

$$L(0,\mu)>N(0,\mu),L'_{s_4}(s_4,\mu)<N'_{s_4}(s_4,\mu),L(\overline{s}_4,\mu)-N(\overline{s}_4,\mu)=-w_4^{33}\delta\overline{s}_4^{\alpha}<0,$$

where $\overline{s}_4=\dfrac{w_4M_4^1\mu}{w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)}$.

When $\left|M_4^1\mu\right|=o\left(\left|w_2^{31}M_2^3\mu+M_2^1\mu\right|\right)$, $0<\overline{s}_4\ll1$. It is clear that (3.2) has a unique solution s_4^{1*} satisfying $0<s_4^{1*}<\overline{s}_4\ll1$. Putting it into the second equation of (3.1), there is $w_4^{13}\delta\left(s_4^{1*}\right)^{\frac{\lambda_2^2}{\lambda_2^1}}+w_4^{11}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)s_4^{1*}+w_4M_4^3\mu+h.o.t.=0$, it defines a surface

$$L_4^1(\mu)=\left\{\mu:w_4^{13}\delta\left(\frac{w_4M_4^1\mu}{w_2^{31}w_4^{31}M_2^3\mu+w_4^{31}M_2^1\mu}\right)^{\frac{\lambda_2^2}{\lambda_2^1}}+w_4^{11}\left(w_4^{31}\right)^{-1}w_4M_4^1\mu+w_4M_4^3\mu+h.o.t.=0\right\},$$

with a normal surface $\Sigma=span\left\{M_4^3\right\}$ at $\mu=0$ for $M_4^3\mu>0$. That is to say, system (1.6) has the only one heteroclinic orbit $\gamma_1(t)$ consisting of p_1 and p_3 near $\Gamma_1\cup\Gamma_3$ for $\mu\in L_4^1(\mu)$.

3) If $w_4^{33}>0$, $\mu\in B_+^+$ or $w_4^{33}<0$, $\mu\in B_-^-$, there are two special cases:

a) As $\left|\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right|s_4\ll\max\left\{s_4^{\alpha},\left|M_4^1\mu\right|\right\}$, Equation (3.2) can be simplified to be

$$w_4M_4^1\mu+w_4^{33}\delta\overline{s}_4^{\alpha}+h.o.t.=0.\tag{3.3}$$

It has a solution $s_4^{2*}=\left(-\dfrac{w_4M_4^1\mu}{w_4^{33}\delta}\right)^{\frac{1}{\alpha}}+h.o.t.$. Substituting s_4^{2*} into the second equation of (2.11), we get immediately a surface $L_4^2(\mu)$ tangent to $L_{14}(\mu)$,

$$L_4^2(\mu)=\left\{\mu: w_4^{11}\left(-\frac{w_4M_4^1\mu}{w_4^{33}\delta}\right)^{\frac{\lambda_2^1}{\lambda_2^2}}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)-w_4^{13}\left(w_4^{33}\right)^{-1}w_4M_4^1\mu+w_4M_4^3\mu+h.o.t.=0\right\}$$

for $w_4M_4^3\mu>0$. So system (1.6) has a heteroclinic orbit consisting of p_3 and p_1 near $\Gamma_1\cup\Gamma_3$ for $\mu\in L_4^2(\mu)$. Next putting the expression of s_4^{2*} into the verification condition, it is equivalently $\left|\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right|\ll\left|M_4^1\mu\right|^{1-\frac{1}{\alpha}}$.

b) As $\left|M_4^1\mu\right|\ll\max\left\{s_4^\alpha,\left|\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right|s_4\right\}$, Equation (3.2) is then

$$w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)s_4-w_4^{33}\delta s_4^\alpha+h.o.t.=0, \tag{3.4}$$

there is a small positive solution $s_4^{3*}=\left(\frac{w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)}{w_4^{33}\delta}\right)^{\frac{1}{\alpha-1}}$. In the same way, we can get the surface $L_4^3(\mu)$ which is tangent to L_{14} with the condition $\left|M_4^1\mu\right|\gg\left|\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right|^{1-\frac{1}{\alpha}}$, where

$$L_4^3(\mu)=\left\{\mu: w_4^{13}\delta\left(w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}}+\left(w_4^{33}\delta\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}}w_4M_4^3\mu+h.o.t.=0\right\}$$

So system (1.6) has a heteroclinic orbit consisting of p_3 and p_1 in the region $\Gamma_1\cup\Gamma_3$ for $\mu\in L_4^3(\mu)$.

4) If $w_4^{33}>0$, $\mu\in B_+^-$ or $w_4^{33}<0$, $\mu\in B_-^+$, without loss of generality, we discuss the case $w_4^{33}>0$, $\mu\in B_+^-$. There are $L(0,\mu)<N(0,\mu)$, $L''_{s_4}(s_4,\mu)-N''_{s_4}(s_4,\mu)=-w_4^{33}\alpha(\alpha-1)\delta s_4^{\alpha-2}+h.o.t.<0$.

Set $L(\tilde{s}_4,\mu)-N(\tilde{s}_4,\mu)=H_4^1(\mu)$, where

$\tilde{s}_4=\left(w_4^{31}\left(w_4^{33}\alpha\delta\right)^{-1}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right)^{\frac{1}{\alpha-1}}+h.o.t.$ is the solution of $L'_{s_4}(s_4,\mu)-N'_{s_4}(s_4,\mu)=0$ and

$$H_4^1(\mu)=w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\left(1-\frac{1}{\alpha}\right)\left(w_4^{31}\left(w_4^{33}\delta\alpha\right)^{-1}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right)^{\frac{1}{\alpha-1}}$$
$$+w_4M_4^1\mu+h.o.t..$$

when $H_4^1(\mu)>0$, the straight line $L(s_4,\mu)$ intersects the curve $N(s_4,\mu)$ exactly at two points $0<s_4^{4*}<\tilde{s}_4<s_4^{5*}$, which means Equation (3.2) has two positive solutions. Therefore, system (1.6) has two heteroclinic orbits connecting p_3 and p_1 near $\Gamma_1\cup\Gamma_3$.

When $H_4^1(\mu)=0$, the equations $L'_{s_4}(s_4,\mu)=N'_{s_4}(s_4,\mu)$ and $L(s_4,\mu)=N(s_4,\mu)$ have the solution $\tilde{s}_4$, therefore the straight line $L(s_4,\mu)$ must be tangent to the curve $N(s_4,\mu)$ at the point $\tilde{s}_4$. Putting it into the second equation of (3.1) yields a surface $L_4^4(\mu)$ with a normal surface $\Sigma=span\left\{M_4^3\right\}$ at $\mu=0$, where

$$L_4^4(\mu)=\left\{\mu: w_4^{13}\delta\left(w_4^{31}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}}+\left(w_4^{33}\alpha\delta\right)^{\frac{\lambda_2^1}{\lambda_2^2-\lambda_2^1}}w_4M_4^3\mu+h.o.t.=0\right\}$$

for $M_4^3\mu > 0$. Then, system (1.6) has a 2-fold heteroclinic orbit connecting p_3 and p_1 near $\Gamma_1 \cup \Gamma_3$.

When $H_4^1(\mu) < 0$, the straight line $L_2(t_2,\mu)$ does not intersect the curve $N_2(t_2,\mu)$ in the half plane, then there is only the transversal heteroclnic orbit $\gamma_3(t)$ connecting p_2 and p_3 near $\Gamma_2 \cup \Gamma_3$.

5) If $\mu \in \{\mu : M_4^1\mu + h.o.t. = M_4^3\mu + h.o.t. = 0\}$, Equation (3.1) is

$$\begin{cases} w_4^{33}\delta s_4^{\frac{\lambda_2^2}{\lambda_2^1}} - w_4^{31}s_4\left(w_2^{31}M_2^3\mu + M_2^1\mu\right) = 0 \\ w_4^{13}\delta s_4^{\frac{\lambda_2^2}{\lambda_2^1}} + w_4^{11}s_4\left(w_2^{31}M_2^3\mu + M_2^1\mu\right) = 0. \end{cases} \tag{3.5}$$

To solve the first equation of (3.5), there is

$$s_4\left(w_4^{33}\delta s_4^{\alpha-1} - w_4^{31}\left(w_2^{31}M_2^3\mu + M_2^1\mu\right)\right) = 0,$$

we can get two solutions $s_4' = 0$ and $s_4'' = \left(\dfrac{w_4^{31}\left(w_2^{31}M_2^3\mu + M_2^1\mu\right)}{w_4^{33}\delta}\right)^{\frac{1}{\alpha-1}}$ for $w_4^{31}w_4^{33}\left(w_2^{31}M_2^3\mu + M_2^1\mu\right) > 0$. However, if $w_4^{31}w_4^{33}\left(w_2^{31}M_2^3\mu + M_2^1\mu\right)$, the above equation has only one zero solution. Equation (3.5) finally defines a surface $\overline{L}_4^3(\mu)$.

Putting the expression s_2'' into the second equation of (3.5) obtains the set of μ as

$$\left\{ \begin{aligned} &\mu \mid w_4^{13}\delta\left(w_4^{33}\delta\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}}\left(w_4^{31}\left(w_2^{31}M_2^3\mu + M_2^1\mu\right)\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}} \\ &+ w_4^{11}\left(w_4^{33}\delta\right)^{\frac{\lambda_2^1}{\lambda_2^2-\lambda_2^1}}\left(w_2^{31}M_2^3\mu + M_2^1\mu\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}} = 0, \left(w_2^{31}M_2^3\mu + M_2^1\mu\right) \neq 0 \end{aligned} \right\},$$

that means the system of (6) coexists two types of heteroclinic orbit: a large-1 heteroclinc orbit connecting with p_3 and p_1, a heteroclinic orbit composed of two orbits which one orbit connects with p_3 and p_2 and the other orbit connects with p_2 and p_1 in the region $\Gamma_1 \cup \Gamma_3$ as $\mu \in \overline{L}_4^3(\mu)$, where

$$\overline{L}_4^3(\mu) = \left\{\mu : M_4^1\mu + h.o.t. = M_4^3\mu + h.o.t. = 0, w_4^{31}w_4^{33}\left(w_2^{31}M_2^3\mu + M_2^1\mu\right) > 0\right\}.$$

Remark 3.1. *The analysis of the third and fourth equations of (20) is similar to the above analysis process, so it will not be repeated here.*

3.2. Bifurcation Conclusions

With the analysis above, we can get the following theorems about existence of the second and the third shape heterodimensional cycle and the large-1 heteroclinic cycle under small perturbation.

Theorem 3.1. *Under (H_1)-(H_4) and Rank $\left(M_1^1, M_1^3, M_4^1, M_4^3\right) \geq 3$, as well as $\mu \in \{\mu : M_4^1\mu + h.o.t. = M_4^3\mu + h.o.t. = 0\}$, there are the following conclusions:*

1) *If $\mu \in \{\mu : w_4^{33} > 0, \mu \in B_-^-\}$ or $\mu \in \{\mu : w_4^{33} < 0, \mu \in B_+^+\}$, the system (1.6) exists the third shape heterodimensional cycle in the $(l-2)$-dimensional surface $L_4^1(\mu)$ with normal vector M_2^1, M_2^3 at $\mu = 0$, where*

$$L_4^1(\mu)=\left\{\mu: M_4^1\mu+h.o.t.=M_4^3\mu+h.o.t.=0, w_4^{31}w_4^{33}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\le 0\right\}.$$

2) *If* $\mu\in\left\{\mu: w_4^{33}>0, \mu\in B_+^-\right\}\left(\mu\in\left\{\mu: w_4^{33}<0, \mu\in B_-^+\right\}\right)$, *the system* (1.6) *exists the third shape heterodimensional cycle near* Γ *as* $\mu\in\tilde{L}_4^4$ *and* $0<|\mu|\ll 1$, *where*

$$\tilde{L}_4^4=\left\{\mu: M_4^1\mu+h.o.t.=M_4^3\mu+h.o.t.=0, w_4^{31}w_4^{33}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\le 0, H_4^1<0\right\}.$$

3) *If* $\mu\in\left\{\mu: w_4^{33}>0, \mu\in B_-^+\right\}$ or $\mu\in\left\{\mu: w_4^{33}<0, \mu\in B_+^-\right\}$, *there exists an* $(l-1)$ *-dimensional surface*

$$\begin{aligned}\hat{L}_4^1=\Big\{&\mu: M_4^1\mu+h.o.t.=M_4^3\mu+h.o.t.=0, w_4^{31}w_4^{33}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\le 0,\\ &\mu\in L_4^1(\mu), \left|M_4^1\mu\right|=o\left(\left|w_2^{31}M_2^3\mu+M_2^1\mu\right|\right)\Big\}\end{aligned}$$

with normal vector M_3^3, M_2^1, M_2^3 *at* $\mu=0$, *which is tangent to the surface* $L_{23}(\mu)$ *at* $\mu=0$, *such that the system* (1.6) *has the second shape hetrodimensional cycle near* Γ *as* $\mu\in\hat{L}_4^1$ *and* $0<|\mu|\ll 1$.

4) *If* $\mu\in\left\{\mu: w_4^{33}>0, \mu\in B_+^+\right\}$ or $\mu\in\left\{\mu: w_4^{33}<0, \mu\in B_-^-\right\}$, *there exist two* $(l-1)$ *-dimensional surfaces*

$$\begin{aligned}\hat{L}_4^2=\Bigg\{&\mu: M_4^1\mu+h.o.t.=M_4^3\mu+h.o.t.=0, w_4^{31}w_4^{33}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\le 0,\\ &\mu\in L_4^2(\mu), \left|\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right|\ll\left|M_4^1\mu\right|^{1-\frac{1}{\alpha}}\Bigg\}\end{aligned}$$

and

$$\begin{aligned}\hat{L}_4^2=\Bigg\{&\mu: M_4^1\mu+h.o.t.=M_4^3\mu+h.o.t.=0, w_4^{31}w_4^{33}\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\le 0,\\ &\mu\in L_4^3(\mu), \left|\left(w_2^{31}M_2^3\mu+M_2^1\mu\right)\right|\gg\left|M_4^1\mu\right|^{1-\frac{1}{\alpha}}\Bigg\}\end{aligned}$$

such that the system (1.6) *has the second shape heterodimensional cycle near* Γ *as* $\mu\in\hat{L}_4^2$, $\mu\in\hat{L}_4^2$, *respectively, and* $0<|\mu|\ll 1$.

An alternative explanation for the existence of the second heterodimensional cycle is as follows. If there is an orbit starting from the section S_1^0 and arriving at the section S_3^1 that passes through the sections S_1^1 and S_3^0 with finite time without orienting to the saddle point p_2, we denote it by $\gamma_1(t,\mu)$. Similarly, we can define $\gamma_4(t,\mu)$ in this way. Set the time of the orbit $\gamma_1(t,\mu)$ from S_1^1 to S_3^0 to be τ_2 and the time of $\gamma_4(t,\mu)$ from S_4^1 to S_2^0 to be τ_4; and from S_k^0 to S_k^1 to be $\tilde{T}_k$, $k=1,2,3,4$, respectively. Moreover, system (1.6) still has solutions $\gamma_j(t,\mu)$, $j=1,4$,

$$\begin{aligned}&\dot{\gamma}_j(t,\mu)=f(\gamma_j,\mu);\\ &\gamma_j(-T_j,\mu)\in S_j^0, \gamma_j(-T_j+\tilde{T}_j,\mu)\in S_j^1,\\ &\gamma_1(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu)\in S_3^1,\\ &\gamma_4(-T_4+\tilde{T}_4+\tau_4+\tilde{T}_2,\mu)\in S_2^1,\end{aligned}$$

$$\begin{aligned}&\left\|\gamma_1\left(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu\right)-\gamma_3^-\left(T_3,\mu\right)\right\|\ll 1,\\&\left\|\gamma_4\left(-T_4-\tilde{T}_4+\tau_4-\tilde{T}_2,\mu\right)-\gamma_2^-\left(T_2,\mu\right)\right\|\ll 1,\end{aligned}\tag{3.6}$$

Theorem 3.2. *Suppose that* (H_1)-(H_4) *hold and Rank* $Rank\left(M_1^1,M_1^3,M_4^1,M_4^3\right)=4$ *there is an* $(l-4)$*-dimensional surface*

$$L_{14}=\left\{\mu:M_1^1\mu+h.o.t.=M_1^3\mu+h.o.t.=M_4^1\mu+h.o.t.=M_4^3\mu+h.o.t.\right\}$$

with a normal plane $\Sigma_{23}=span\left\{M_1^1,M_1^3,M_4^1,M_4^3\right\}$, *such that system* (1.6) *has a unique double heteroclinic loop* ("∞") *in the tubular neighborhood of* Γ *as* $\mu\in L_{24}$, $0<|\mu|\le 1$.

Proof. As we explained above, $s_2=0$ in Equation (2.11) means the flying time of an orbit starting from S_1^1 to S_3^0 is infinite, that is, the orbit must go into the equilibrium p_2 and then leave, which corresponds to a heteroclinic orbit; and for $s_4=0$, it is similar. Hence, set $s_2=s_4=0$ in Equation (2.11), we have

$$\begin{cases}M_1^1\mu+h.o.t.=0,\\M_1^3\mu+h.o.t.=0,\\M_4^1\mu+h.o.t.=0,\\M_4^3\mu+h.o.t.=0.\end{cases}$$

If $\text{Rank}\left(M_1^1,M_1^3,M_4^1,M_4^3\right)=4$, there is a codimension-4 surface with a normal plane spanned by M_1^1,M_1^3,M_4^1,M_4^3 as below

$$L_{14}\left(\mu\right)=\left\{\mu:M_1^1\mu+h.o.t.=M_1^3\mu+h.o.t.=M_4^1\mu+h.o.t.=M_4^3\mu+h.o.t.=0\right\}.$$

when $\mu\in L_{14}$, system (1.6) has four heteroclinic orbits connecting the equilibriums p_i, $i=1,2,3$, and they form an "∞"-type double heterodimensional cycle, or it says that the original heterodimensional cycle is preserved. □

Corresponding, some new orbits $\gamma_k^+(t,\mu)$ (resp. $\gamma_k^-(t,\mu)$) $(k=1,2,3,4)$ appear from unstable (resp. stable) manifold of the equilibrium $p_i\,(i=1,2,3)$ of system (1.6) with the following properties,

$$\begin{aligned}&\dot{\gamma}_k^\pm=f\left(\gamma_k^\pm,\mu\right);\\&\gamma_1^+\left(t,\mu\right)\in W_1^u\left(p_1\right),\quad\gamma_1^-\left(t,\mu\right)\in W_2^s\left(p_2\right),\\&\gamma_2^+\left(t,\mu\right)\in W_2^u\left(p_2\right),\quad\gamma_2^-\left(t,\mu\right)\in W_1^s\left(p_1\right),\\&\gamma_3^+\left(t,\mu\right)\in W_2^u\left(p_2\right),\quad\gamma_3^-\left(t,\mu\right)\in W_3^s\left(p_3\right),\\&\gamma_4^+\left(t,\mu\right)\in W_3^u\left(p_3\right),\quad\gamma_4^-\left(t,\mu\right)\in W_2^s\left(p_2\right),\\&\gamma_k^\pm\left(t,0\right)=\gamma_k\left(t\right),\\&\gamma_k^+\left(-T_k,\mu\right)\in S_k^0,\\&\gamma_k^+\left(-T_k+\tilde{T}_k,\mu\right),\gamma_k^-\left(T_k,\mu\right)\in S_k^1,\\&\left\|\gamma_3^+\left(-T_3+\tilde{T}_3,\mu\right)-\gamma_3^-\left(T_k,\mu\right)\right\|=0\left(k=1,4\right)\\&\left\|\gamma_k^+\left(-T_k+\tilde{T}_k,\mu\right)-\gamma_k^-\left(T_k,\mu\right)\right\|\ll 1\left(k=2,3\right),\end{aligned}\tag{3.7}$$

where $W_i^s\left(p_i\right)$ and $W_i^u\left(p_i\right)$ are the stable and unstable manifolds of the equilibrium p_i, because the original heteroclinic trajectory $\gamma_3(t)$ is obtained as a transversal intersection of 2-dimensional manifolds, which is a structurally stable situation. After a small perturbation, such an intersection is preserved. That is, the gap $\left\|\gamma_3^+\left(-T_3+\tilde{T}_3,\mu\right)-\gamma_3^-\left(T_3,\mu\right)\right\|=0$. As well as, if the gap $\left\|\gamma_k^+\left(-T_k+\tilde{T}_k,\mu\right)-\gamma_k^-\left(T_k,\mu\right)\right\|=0$ $(k=1,2,4)$ in S_k^1, it means that the original double heterodimensional cycles are kept (see Figure 3).

Where $\gamma_k^\pm(t,\mu)$ $(k=2,4)$, and $\gamma_3^-(t,\mu)$ still meet Equation (3.1). Clearly system (1.6) has the second shape heterodimensional cycle, if the gaps $\left\|\gamma_1\left(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu\right)-\gamma_3^-\left(T_3,\mu\right)\right\|=0$, $\left\|\gamma_2^-\left(-T_2,\mu\right)-\gamma_2^+\left(T_2+\tilde{T}_2,\mu\right)\right\|=0$ (see Figure 4).

Remark 3.2. *The second heterodimensional cycle consists of two saddles of* (1.2) *type and one saddle of* (2.1) *type and is composed of one big orbit linking* p_1, p_3 *and two orbits linking* p_3, p_2 *and* p_2, p_1 *respectively* (*see* Figure 3).

Remark 3.3. *As for the other theorem of the similar second shape heterdimensional cycle which consists of two saddles of* (1.2) *type and one saddle of* (2.1) *type and is composed of one big orbit linking* p_1, p_3 *and two orbits linking* p_3, p_2 *and* p_2, p_1 *respectively is analogous to theorem* 3.1, *so it will not be repeated here.*

Theorem 3.3. *Suppose* (H_1)-(H_4) *are valid and* $\mathrm{Rank}\left(M_1^1, M_1^3, M_4^1, M_4^3\right)\geq 3$, *there are the following conclusions*:

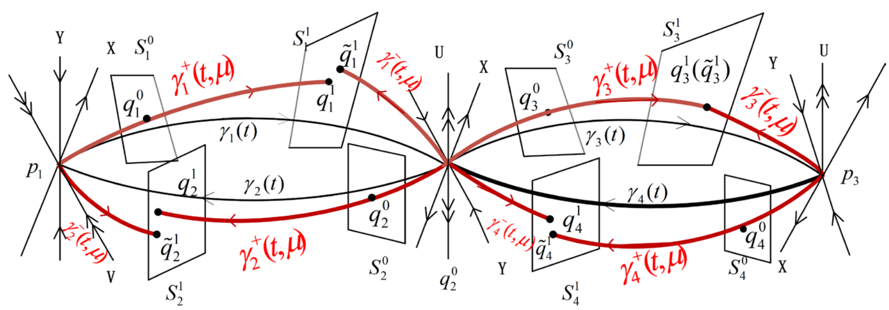

Figure 3. The gap $\left\|\gamma_k^+\left(-T_k+\tilde{T}_k,\mu\right)-\gamma_k^-\left(T_k,\mu\right)\right\|=0\,(k=1,2,4)$ in the figure, the original double heterodimensional cycles exists.

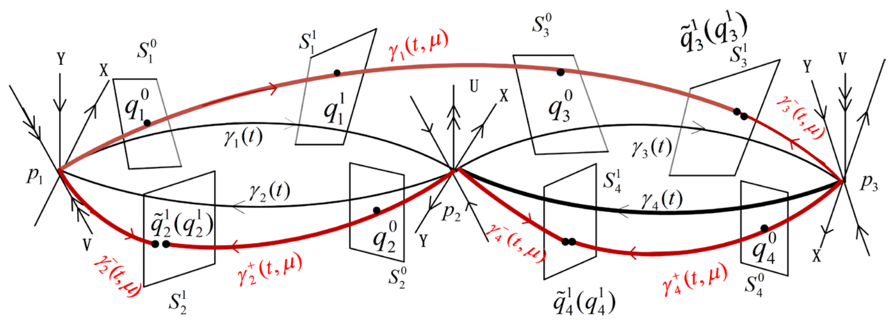

Figure 4. The gap $\left\|\gamma_1\left(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu\right)-\gamma_3^-\left(-T_3,\mu\right)\right\|=0$, $\left\|\gamma_2^-\left(-T_2,\mu\right)-\gamma_2^+\left(T_2+\tilde{T}_2,\mu\right)\right\|=0$ in the figure, there is the second heterodimensional cycle.

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1) *If* $\mu \in \{\mu : w_1^{33} > 0, w_4^{33} > 0, \mu \in B_-^+, \mu \in N_-^+\}$ *or* $\mu \in \{\mu : w_1^{33} < 0, w_4^{33} < 0, \mu \in B_+^-, \mu \in N_+^-\}$, *there exists an* $(l-2)$*-dimensional surface*

$$H_{24}^1(\mu) = \Big\{\mu : |M_4^1\mu| = o\left(\left|w_2^{31}M_2^3\mu + M_2^1\mu\right|\right), |M_1^1\mu| = o\left(\left|w_3^{32}M_3^3\mu + M_3^1\mu\right|\right),$$
$$\mu \in L_2^1(\mu) \cap L_4^1(\mu), L_2^1(\mu) \cap L_4^1(\mu) \neq \varnothing\Big\}$$

with normal vector M_1^1, M_4^1 *at* $\mu = 0$, *which is tangent to the surface* $L_{14}(\mu)$, *then the system* (1.6) *has a* 1*-fold large-*1 *heteroclinic cycle near* Γ *as* $\mu \in H_{24}^1(\mu)$ *and* $0 < |\mu| \ll 1$, *where*

$$L_2^1(\mu) = \left\{\mu : w_1^{13}\delta\left(\frac{w_1 M_1^1\mu}{w_1^{31}w_3^{32}M_3^3\mu + w_1^{31}M_3^1\mu}\right)^{\frac{\lambda_2^2}{\lambda_2^1}} - w_1^{11}\left(w_1^{31}\right)^{-1} w_1 M_1^1\mu + w_1 M_1^3\mu + h.o.t. = 0\right\}.$$

2) *If* $\mu \in \{\mu : w_1^{33} > 0, w_4^{33} > 0, \mu \in B_-^+, \mu \in N_-^+\}$ *or* $\mu \in \{\mu : w_1^{33} < 0, w_4^{33} < 0, \mu \in B_+^-, \mu \in N_+^-\}$, *there exists two* $(l-2)$*-dimensional surface*

$$H_{24}^2(\mu) = \left\{\mu : |M_4^1\mu| \ll \left|\left(w_2^{31}M_2^3\mu + M_2^1\mu\right)\right|^{1-\frac{1}{\alpha}}, |M_1^1\mu| \ll \left|\left(w_3^{32}M_3^3\mu + M_3^1\mu\right)\right|^{1-\frac{1}{\alpha}},\right.$$
$$\left.\mu \in L_2^2(\mu) \cap L_4^2(\mu), L_2^2(\mu) \cap L_4^2(\mu) \neq \varnothing\right\}$$

and

$$H_{24}^3(\mu) = \left\{\mu : \left|\left(w_2^{31}M_2^3\mu + M_2^1\mu\right)\right|^{1-\frac{1}{\alpha}} \ll |M_4^1\mu|, \left|\left(w_3^{32}M_3^3\mu + M_3^1\mu\right)\right|^{1-\frac{1}{\alpha}} \ll |M_1^1\mu|,\right.$$
$$\left.\mu \in L_2^3(\mu) \cap L_4^3(\mu), L_2^3(\mu) \cap L_4^3(\mu) \neq \varnothing\right\}$$

both with normal vector M_1^3, M_4^3 *at* $\mu = 0$, *which both are tangent to the surface* $L_{23}(\mu)$, *then the system* (1.6) *has a* 1*-fold large-*1 *heteroclinic cycle near* Γ *as* $\mu \in H_{24}^2(\mu)$, $\mu \in H_{24}^3(\mu)$, *respectively, and* $0 < |\mu| \ll 1$, *where*

$$L_2^2(\mu) = \left\{\mu : w_1^{11}\left(-\frac{w_1 M_1^1\mu}{w_1^{33}\delta}\right)^{\frac{\lambda_2^1}{\lambda_2^2}}\left(w_3^{32}M_3^3\mu + M_3^1\mu\right) + w_1^{13}\left(w_1^{33}\right)^{-1} w_1 M_1^1\mu - w_1 M_1^3\mu + h.o.t. = 0\right\}$$

and

$$L_2^3(\mu) = \left\{\mu : w_1^{13}\delta\left(w_1^{11}\left(w_3^{32}M_3^3\mu + M_3^1\mu\right)\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}} - \left(w_1^{33}\delta\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}} w_1 M_1^3\mu + h.o.t. = 0\right\}.$$

3) *If* $\mu \in \{\mu : w_1^{33} > 0, \mu \in B_+^-, w_4^{33} > 0, \mu \in N_+^-\}$, *and* $H_2^1 > 0, H_4^1 > 0$, *the system* (1.6) *has two* 1*-fold large-*1 *heteroclinic cycles near* Γ, *where*

$$H_2^1(\mu) = w_1^{11}\left(w_3^{32}M_3^3\mu + M_3^1\mu\right)\left(1 - \frac{1}{\alpha}\right)\left(w_1^{11}\left(w_1^{33}\delta\alpha\right)^{-1}\left(w_3^{32}M_3^3\mu + M_3^1\mu\right)\right)^{\frac{1}{\alpha-1}}$$
$$+ w_1 M_1^1\mu + h.o.t..$$

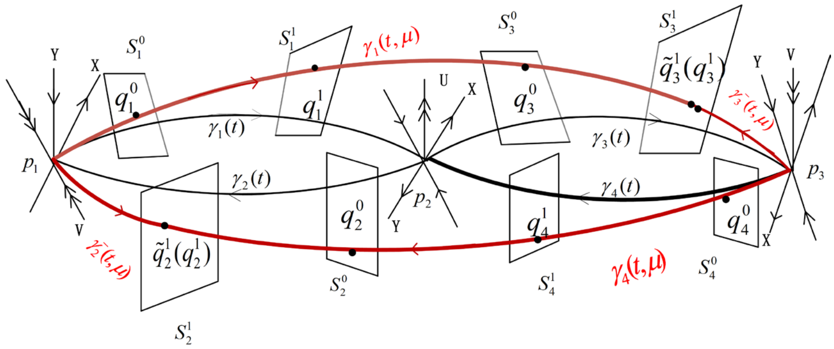

Figure 5. The gap $\left\|\gamma_1\left(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu\right)-\gamma_3^-\left(T_3,\mu\right)\right\|=0$, $\left\|\gamma_4\left(-T_4+\tilde{T}_4+\tau_4+\tilde{T}_2,\mu\right)-\gamma_2^-\left(T_2,\mu\right)\right\|=0$ in the figure, there is the large 1-heteroclinic cycle.

4) *If* $w_1^{33}>0, w_4^{33}>0, \mu\in B_-^+, \mu\in N_-^+$, *there exists a* $(l-2)$*-dimensional surface* H_{24}^6 *with normal vector* M_1^3, M_4^3, *which is tangent to the surface* $L_{14}(\mu)$ *at* $\mu=0$, *where*

$$H_{24}^4(\mu)=\left\{\mu: H_2^1=0, H_4^1=0, \mu\in L_2^4\cap L_4^4, L_2^4\cap L_4^4\neq\varnothing\right\},$$

then the system (1.6) *has one* 2*-fold large-*1 *heteroclinic cycles near* Γ *for* $\mu\in H_{24}^4$, *where*

$$L_2^4(\mu)=\left\{\mu: w_1^{13}\delta\left(w_1^{11}\left(w_3^{32}M_3^3\mu+M_3^1\mu\right)\right)^{\frac{\lambda_2^2}{\lambda_2^2-\lambda_2^1}}+\left(w_1^{33}\alpha\delta\right)^{\frac{\lambda_2^1}{\lambda_2^2-\lambda_2^1}}w_1M_1^3\mu+h.o.t.=0\right\}.$$

For the alternative explanation from the gaps for the existence of the large-1 heteroclinic cycle is the following. If $\gamma_k(t,\mu)(k=1,4)$, and $\gamma_k^-(t,\mu)(k=2,3)$ still meet Equation (3.2) and (3.1). Clearly system (1.6) has a large 1-heteroclinic cycle composed of two big orbits linking p_1,p_3 and p_3,p_1 of (1.2) type respectively, if the gaps $\left\|\gamma_1\left(-T_1+\tilde{T}_1+\tau_2+\tilde{T}_3,\mu\right)-\gamma_3^-\left(T_3,\mu\right)\right\|=0$, $\left\|\gamma_4\left(-T_4-\tilde{T}_4+\tau_4-\tilde{T}_2,\mu\right)-\gamma_2^-\left(T_2,\mu\right)\right\|=0$ (see Figure 5).

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Conflicts of Interest

The authors declare that they have no competing interests.

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